Optimal Glideslope Guidance for Spacecraft Rendezvous

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I. Introduction

Many space applications involve rendezvous with a vehicle in circular orbit. A subset of these applications requires the visiting vehicle to approach with a constant direction as seen by the target. This is the case for vehicles approaching the International Space Station (ISS), for example. By approaching it in a straight line the crew onboard the station can easily monitor non-nominal situations. The Space Shuttle employs a straight-line guidance law called glideslope [1]. Vehicles visiting the ISS usually employ a fixed direction terminal approach, including HTV [2], ATV [3], and Cygnus [4]. In this work a constant direction guidance law is developed to rendezvous a target in circular orbit. This type of trajectory is referred to as glideslope.

Much work exist in the general area of optimal space trajectories, an illustrative early work is that by Lawden [5]. Carter studied minimum delta-v maneuvers to rendezvous with a vehicle in circular orbit [6]. The approach used by Carter and by many authors after him is to optimize the system subject to the linearized dynamics, the so-called Clohessy-Wiltshire equations [7]. The rendezvous strategy by Lembeck and Prussing [8] is to add to an initial impulsive phase a low-thrust phase. Since continuous thrust is necessary to guide on the glideslope, this work also assumes low-thrust propulsion.

Various aspects of this problem were generalized. Carter and Humi study the impulsive rendezvous in proximity of a general Keplerian orbit [9] while Carter studies the continuous thrust

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case [10]. Power limitations and thrust bounds are also studied [11]. Guelman and Aleshin [12] develop a two-stage solution for the fixed terminal-approach direction. The first stage consists in an unconstrained optimization that puts the vehicle on the glideslope. The second stage is along the glideslope. In this work only the terminal phase is considered, when the spacecraft is required to fly on the glideslope.

The current work differs considerably from the work of Guelman and Aleshin. In their work the constraint is not enforced directly, but the squared distance to the glideslope is added to the performance index with a weighting parameter. The bigger the parameter the closer the constraint is to be satisfied. This work's approach is to satisfy the constraint exactly. Another difference between the two works is that Guelman and Aleshin solve their optimization numerically, while a closed-form solution is presented in this paper. The optimal guidance solution applies when the vehicle is on the glideslope. In practice an inner loop controller is needed to maintain the vehicle on the desired terminal direction.

II. Optimal Guidance

Clohessy-Wiltshire (CW) equations are used to express the dynamics of the chaser vehicle in proximity of a target in circular orbit. The coordinate frame used in this derivation is centered at the target, has the x-axis along the negative velocity vector and the y-axis along the radial direction. In this coordinate system the linearized equations of relative motion are given by

$$\ddot{x} = 2\omega\dot{y} + u_x \tag{1}$$

$$\ddot{y} = 3\omega^2 y - 2\omega\dot{x} + u_y \tag{2}$$

$$\ddot{z} = -\omega^2 z + u_z. \tag{3}$$

The control acceleration is given by $\mathbf{u} = [u_x \;\; u_y \;\; u_z]^{\mathrm{T}}$, and ω is the orbital angular velocity of the target.

For a non-impulsive, power limited propulsion system, an appropriate performance index is (see for example [13] and citations therein)

$$\mathcal{J} = \frac{1}{2}\int_0^{t_f} \mathbf{u}^{\mathrm{T}}\mathbf{u}\, dt, \tag{4}$$

subject to the linear dynamics governed by the CW equations and constrained to be on the glideslope. The performance index of Eq. (4) is particularly useful for continuous low-thrust systems; for impulsive maneuvers a minimum delta-v solution is more appropriate. To remain on the glideslope at all times is necessary to continuously thrust in the direction perpendicular to the glideslope. For electric propulsion systems Eq. (4) is also a minimum fuel solution. For any thruster the expelled mass is given by

$$\dot{m}_e = \frac{T}{I_{sp} g_0}, \tag{5}$$

where T is the desired thrust magnitude and g_0 is constant. For an electric thruster the specific impulse is given by

$$I_{sp} = \frac{2P}{g_0 T}, \tag{6}$$

where P is the output power. The input electrical power is greater than P and depends on the efficiency of the thruster. While the output power is not always constant with a throttleable engine, for the purpose of this work it is assumed it is. By combining the last two equations it results that the total expelled mass is proportional to the square of the thrust

$$\dot{m}_e = \frac{T^2}{2P}. \tag{7}$$

For low thrust vehicles when the expelled mass is negligible with respect to the total mass it follows that the expelled mass is proportional to the square of the commanded acceleration.

The glideslope angle θ is defined as the angle between the direction of the approach and the positive x-axis, counted positive using the right-hand rule around the positive z-axis. The direction along the line of approach is called radial r, and its in-plane perpendicular is called transversal t.

$$\begin{bmatrix} r \\ t \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{8}$$

in this frame the equations of motion become

$$\ddot{r} = 2\omega\dot{t} + 3\omega^2 r \sin^2\theta + 3\omega^2 t \sin\theta\cos\theta + u_r \tag{9}$$

$$\ddot{t} = -2\omega\dot{r} + 3\omega^2 r \sin\theta\cos\theta + 3\omega^2 t \cos^2\theta + u_t \tag{10}$$

$$\ddot{z} = -\omega^2 z + u_z. \tag{11}$$

In order to approach the target in a straight line it is necessary that $t = \dot{t} = \ddot{t} = 0$, hence

$$\ddot{r} = 3\omega^2 r \sin^2\theta + u_r \tag{12}$$

$$0 = -2\omega\dot{r} + 3\omega^2 r \sin\theta\cos\theta + u_t \tag{13}$$

$$\ddot{z} = -\omega^2 z + u_z, \tag{14}$$

it follows that the optimal transversal acceleration is given by

$$u_t^* = 2\omega\dot{r} - 3\omega^2 r \sin\theta\cos\theta. \tag{15}$$

It is assumed that the desired approach direction lies in the plane of motion of the target's vehicle. Most vehicles employ the strategy of eliminating the out-of-plane component early in the rendezvous phase. Therefore the performance index to be minimized becomes

$$\mathcal{J} = \frac{1}{2}\int_0^{t_f} [u_r^2 + u_t^2]dt = \frac{1}{2}\int_0^{t_f} [u_r^2 + (2\omega\dot{r} - 3\omega^2 r\sin\theta\cos\theta)^2]dt \tag{16}$$

subject to the kinematic constraint

$$\dot{r} = v \tag{17}$$

$$\dot{v} = 3\omega^2 r \sin^2\theta + u_r, \tag{18}$$

and the boundary conditions

$$r(0) = r_0 \qquad r(t_f) = r_f \qquad v(0) = v_0 \qquad v(t_f) = v_f. \tag{19}$$

The Hamiltonian is given by

$$H = \frac{1}{2}[u_r^2 + (2\omega\dot{r} - 3\omega^2 r\sin\theta\cos\theta)^2] + \lambda_r v + \lambda_v(3\omega^2 r\sin^2\theta + u_r), \tag{20}$$

the costate equations are given by

$$\dot{\lambda}_r = -\frac{\partial H}{\partial r} = -(2\omega\dot{r} - 3\omega^2 r\sin\theta\cos\theta)(-3\omega^2\sin\theta\cos\theta) - 3\omega^2\sin^2\theta\lambda_v \tag{21}$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -2\omega(2\omega\dot{r} - 3\omega^2 r\sin\theta\cos\theta) - \lambda_r \tag{22}$$

and the control optimality condition is given by

$$\frac{\partial H}{\partial u_r} = 0 = u_r + \lambda_v. \tag{23}$$

Therefore the optimal control is given by $u_r = -\lambda_v$ and augmenting states and costates in a single vector it follows that

$$\frac{d}{dt}\begin{bmatrix} r \\ v \\ \lambda_r \\ \lambda_v \end{bmatrix} = \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{24}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 \sin^2\theta & 0 & 0 & -1 \\ -9\omega^4 \sin^2\theta\cos^2\theta & 6\omega^3 \sin\theta\cos\theta & 0 & -3\omega^2\sin^2\theta \\ 6\omega^3\sin\theta\cos\theta & -4\omega^2 & -1 & 0 \end{bmatrix}. \tag{25}$$

Partitioning the state transition matrix of $\mathbf{A}$ in four 2-by-2 blocks

$$\mathbf{\Phi}_{\mathbf{A}}(\tau, t_0) = e^{\mathbf{A}(\tau - t_0)} = \begin{bmatrix} \mathbf{\Phi}_{rr}(\tau - t_0) & \mathbf{\Phi}_{r\lambda}(\tau - t_0) \\ \mathbf{\Phi}_{\lambda r}(\tau - t_0) & \mathbf{\Phi}_{\lambda\lambda}(\tau - t_0) \end{bmatrix} \tag{26}$$

the initial values of the costates are determined to be

$$\begin{bmatrix} \lambda_r(t_0) \\ \lambda_v(t_0) \end{bmatrix} = \mathbf{\Phi}_{r\lambda}^{-1}(t_f - t_0)\left(\begin{bmatrix} r_f \\ v_f \end{bmatrix} - \mathbf{\Phi}_{rr}(t_f - t_0)\begin{bmatrix} r_0 \\ v_0 \end{bmatrix}\right), \tag{27}$$

and the optimal control history is given by

$$u_r^*(t) = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{\Phi}_{\mathbf{A}}(t, t_0)\,\mathbf{x}(t_0). \tag{28}$$

To show that the solution is a minimum the Weierstrass and Legendre-Clebsch conditions are tested [14]. The Weierstrass condition requires that the Hamiltonian evaluated at any admissible comparison control u_r is larger than the Hamiltonian evaluated at the optimal control u_r^*. From Eq. (20)

$$H(u_r) - H(u_r^*) = [0.5u_r^2 + \lambda_v u_r - 0.5(u_r^*)^2 - \lambda_v u_r^*], \tag{29}$$

substituting $\lambda_v = -u_r^*$

$$H(u_r) - H(u_r^*) = \frac{1}{2}(u_r - u_r^*)^2 \geq 0, \tag{30}$$

therefore the Weierstrass condition is satisfied.

The Legendre-Clebsch condition requires that the second order partial of the Hamiltonian with respect to the optimal control is positive definite. From Eq. (20)

$$\frac{\partial^2 H}{\partial u_r^2} = 1, \tag{31}$$

hence the Legendre-Clebsch condition is also satisfied and the solution is indeed a minimum.

III. Implementation Considerations

The most computationally demanding part of the algorithm is the computation of the matrix exponential in order to obtain the state transition matrix. However applying the Cayley-Hamilton theorem vastly reduces the complexity. The matrix exponential of the 4-by-4 matrix $\mathbf{A}$ can be computed as

$$e^{\mathbf{A}\Delta t} = \sum_{k=0}^{3} \alpha_k \mathbf{A}^k, \tag{32}$$

where the coefficients α_k are calculated solving

$$e^{\lambda_i \Delta t} = \sum_{k=0}^{3} \alpha_k \lambda_i^k, \tag{33}$$

where λ_i is the i-th eigenvalue of $\mathbf{A}$.

Since $\mathbf{A}$ is constant, so are its four eigenvalues, which can be computed *a priori* and are given by

$$\lambda_i = \pm \left(3\sin^2\theta + 2 \pm \sqrt{9\sin^4\theta + 3\sin^2\theta + 4}\right)^{1/2} \omega. \tag{34}$$

Using vector notation

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}^{\mathrm{T}} \qquad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^{\mathrm{T}} \tag{35}$$

it follows that

$$\boldsymbol{\alpha} = \boldsymbol{\Lambda}^{-1} e^{\boldsymbol{\lambda}\Delta t}, \tag{36}$$

where $e^{\mathbf{w}}$ represents a vector whose components are the exponentials of the components of vector $\mathbf{w}$. The four columns of $\boldsymbol{\Lambda}$ are given by the eigenvalues elevated to the zero-th, first, second, and third power, respectively. The inverse of $\mathbf{\Lambda}$ needs to be computed only once and can be a parameter uploaded from the ground.

A practical implementation should not assume that t, z and their derivatives are always zero but dispersions should be corrected. An inner loop controller needs to be implemented to cancel the out-of-plane and transversal components. Alternatively the control thrust can be chosen using a simple PD controller

$$u_r = u_r^* - 2\omega\dot{t} - 3\omega^2 \sin\theta\cos\theta t \tag{37}$$

$$u_t = u_t^* - 3\omega^2 \cos^2\theta t - k_p t - k_d \dot{t} \tag{38}$$

$$u_z = -k_z \dot{z}. \tag{39}$$

where the asterisk represents the previously defined values from the guidance law. The positive coefficients k_p, k_d, and k_z are design parameters to dampen the dispersions.

IV. Three Special Cases

Three scenarios deserve special attention, these cases are the common glideslope angles that posses an analytic solution. These cases are the V-bar approach ($\theta = \pi$) in which the chaser starts directly in front of the target, the minus V-bar approach ($\theta = 0$) in which the chaser starts directly behind the target, and the R-bar approach ($\theta = -\pi/2$) in which the chaser starts directly below the target. A minus R-bar approach also possesses an analytical solution, but in practice this kind of approach is not used.

The V-bar approach is the most common Space Shuttle rendezvous strategy. Eq. (34) shows that $\mathbf{A}$ has repeated eigenvalues only when $\sin\theta = 0$. Under this circumstance the repeated eigenvalues are equal to zero and the other two are given by $\pm 2\omega$. The system of Eq. (33) is not solvable in this situation, the equation relative to the repeated eigenvalue needs to be replaced by its derivative

$$\lambda_i\, e^{\lambda_i \Delta t} = \sum_{k=1}^{3} \alpha_k \lambda_i^{k-1}. \tag{40}$$

Solving this modified set of equations results in

$$\alpha_0 = 1 \tag{41}$$

$$\alpha_1 = \Delta t \tag{42}$$

$$\alpha_2 = \frac{\cosh(2\omega\Delta t) - 1}{4\omega^2} \tag{43}$$

$$\alpha_3 = \frac{\sinh(2\omega\Delta t) - 2\omega\Delta t}{8\omega^3}. \tag{44}$$

Matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4\omega^2 & -1 & 0 \end{bmatrix} \tag{45}$$

and its state transition matrix is given by

$$\mathbf{\Phi}(\Delta t) = \begin{bmatrix} 1 & \Delta t + 4\omega^2\alpha_3 & \alpha_3 & -\alpha_2 \\ 0 & 1 + 4\omega^2\alpha_2 & \alpha_2 & -\Delta t - 4\omega^2\alpha_3 \\ 0 & 0 & 1 & 0 \\ 0 & -4\omega^2\Delta t - 16\omega^4\alpha_3 & -\Delta t - 4\omega^2\alpha_3 & 1 + 4\omega^2\alpha_2 \end{bmatrix}. \tag{46}$$

The state transition matrix for the minus V-bar approach (which is used by ATV and some Russian vehicles) is also given by Eq. (46). To implement the algorithm the inverse of the top-right component of $\mathbf{\Phi}(\Delta t)$ is also needed, for the V-bar approach this inverse is given by

$$\mathbf{\Phi}_{r\lambda}^{-1}(\Delta t) = \frac{1}{-\alpha_3(\Delta t + 4\omega^2\alpha_3) + \alpha_2^2} \begin{bmatrix} -\Delta t - 4\omega^2\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 \end{bmatrix}. \tag{47}$$

The R-bar approach consists in going to the ISS from below and it is used by HTV. Matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -3\omega^2 \\ 0 & -4\omega^2 & -1 & 0 \end{bmatrix}, \tag{48}$$

its eigenvalues are $\pm\omega$ and $\pm 3\omega$, the coefficients are given by

$$\alpha_0 = \frac{-\cosh(3\omega t) + 9\cosh(\omega t)}{8} \tag{49}$$

$$\alpha_1 = \frac{-(1/3)\sinh(3\omega t) + 9\sinh(\omega t)}{8\omega} \tag{50}$$

$$\alpha_2 = \frac{\cosh(3\omega t) - \cosh(\omega t)}{8\omega^2} \tag{51}$$

$$\alpha_3 = \frac{\sinh(3\omega t) - 3\sinh(\omega t)}{24\omega^3} \tag{52}$$

and the state transition matrix is

$$\mathbf{\Phi}(\Delta t) = \begin{bmatrix} \alpha_0 + 3\omega^2\alpha_2 & \alpha_1 + 7\omega^2\alpha_3 & \alpha_3 & -\alpha_2 \\ 3\omega^2\alpha_1 + 21\omega^4\alpha_3 & \alpha_0 + 7\omega^2\alpha_2 & \alpha_2 & -\alpha_1 - 10\omega^2\alpha_3 \\ 36\omega^6\alpha_3 & 12\omega^4\alpha_2 & \alpha_0 + 3\omega^2\alpha_2 & -3\omega^2\alpha_1 - 21\omega^4\alpha_3 \\ -12\omega^4\alpha_2 & -4\omega^2\alpha_1 - 40\omega^4\alpha_3 & -\alpha_1 - 7\omega^2\alpha_3 & \alpha_0 + 7\omega^2\alpha_2 \end{bmatrix}. \tag{53}$$

Finally the inverse of the top right component of the state transition matrix is given by

$$\mathbf{\Phi}_{r\lambda}^{-1}(\Delta t) = \frac{1}{-\alpha_3(\Delta t + 10\omega^2\alpha_3) + \alpha_2^2} \begin{bmatrix} -\alpha_1 - 10\omega^2\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 \end{bmatrix}. \tag{54}$$

V. Numerical Examples

In this section numerical examples are presented to assess the validity of the guidance law. In the examples below the simulation uses nonlinear dynamics assuming central gravity only.

The first numerical example is a V-bar approach with glideslope angle $\theta = \pi$. The chaser vehicle is placed 200 meters in front of the target. Fig. 1 shows the relative trajectory. The chaser starts in front of the target at the minus 200 meters x coordinate. The final position is at the origin. Fig. 2 shows the radial and transversal components of the relative velocity. Fig. 3 shows the commanded acceleration.

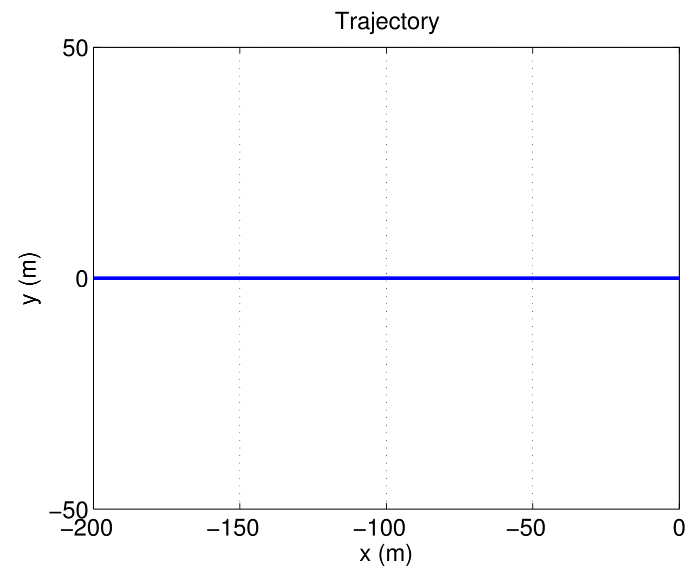

Fig. 1 V-bar approach trajectory.

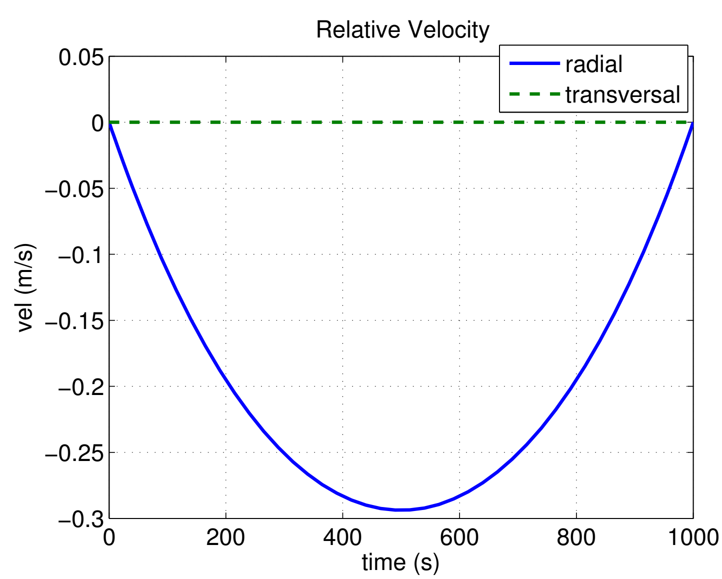

Fig. 2 Velocity Components in V-bar approach.

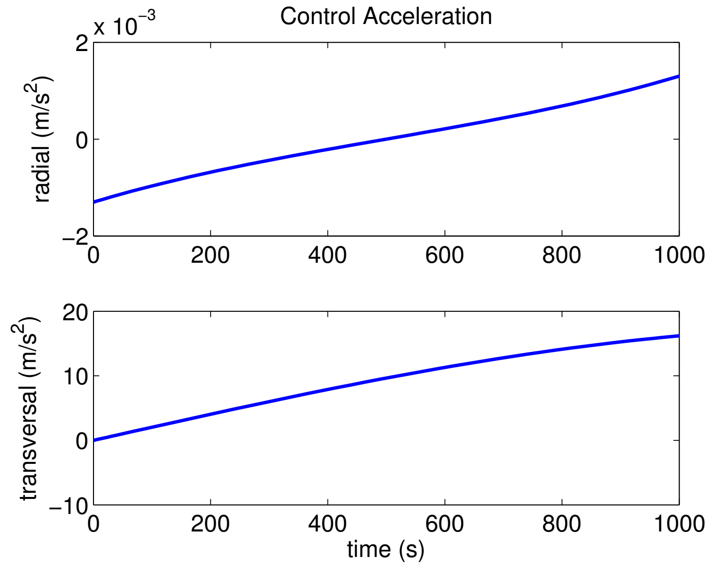

Fig. 3 Commanded Acceleration in V-bar approach.

The second example is an R-bar approach. The chaser vehicle is placed 200 meters below the target with an offset of 10 meters from the glideslope. The control gains in this example are chosen as $k_p = 5E - 4$ and $k_d = 1E - 2$. Fig. 4 shows the relative trajectory. This initial position is below the target at the minus 200 meters y coordinate. The final position is again at the origin. Fig. 5 shows the radial and transversal components of the relative velocity while Fig. 6 shows the commanded optimal acceleration.

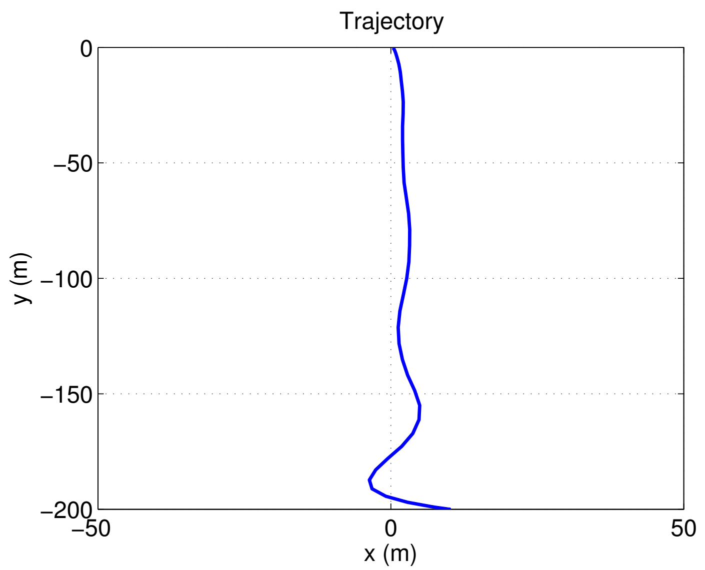

Fig. 4 R-bar approach with initial off-glideslope error.

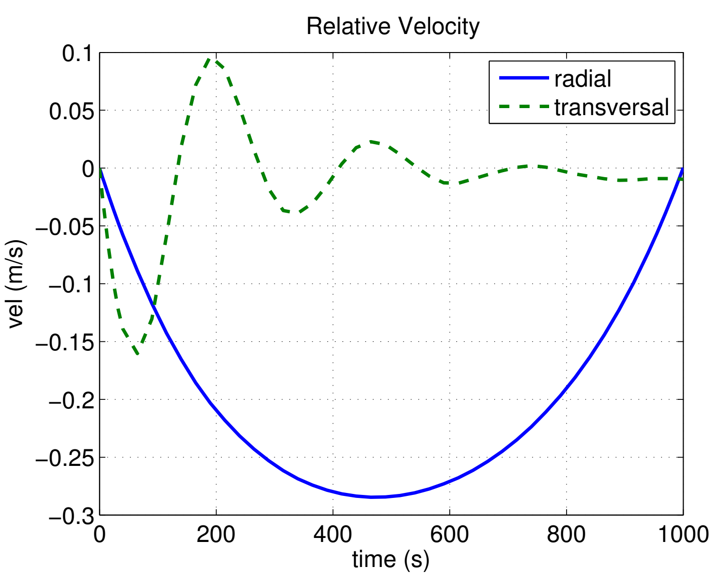

Fig. 5 Velocity Components in R-bar approach.

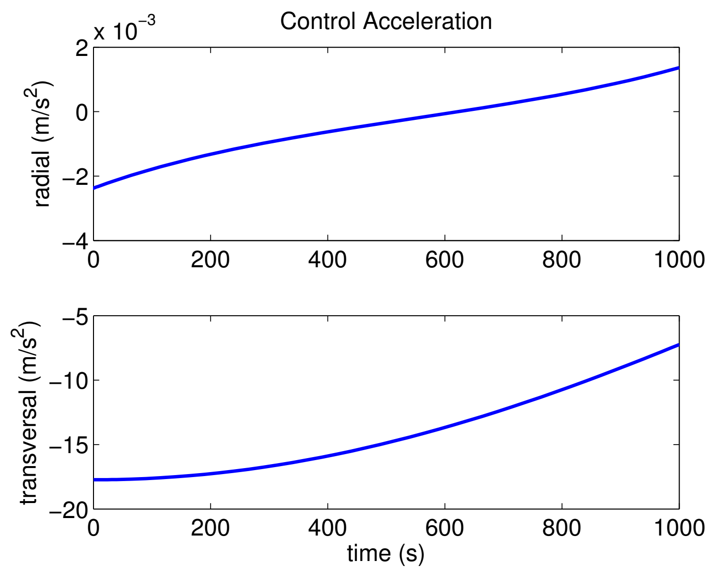

Fig. 6 Commanded Acceleration in R-bar approach.

VI. Conclusions

A common approach for the final phase of spacecraft rendezvous is to approach the target along a straight line, the so-called glideslope. In this paper a new fixed terminal direction guidance law to rendezvous with a target vehicle in circular orbit is introduced. The guidance law is derived by minimizing a commonly employed performance index that assumes finite thrust and is particularly adapt for electric thrusters. The guidance law is provided in closed-form assuming the vehicle starts from the glideslope. Calculation of a matrix exponential is required to compute the optimal acceleration. By performing some of the calculations *a priori* the total computational cost can be greatly reduced. For some commonly-employed glideslope angles the matrix exponential can be written analytically which allows for further reduction of the onboard computations.

The guidance law is obtained in closed-form by employing linearized dynamics, numerical examples demonstrate the validity of this assumption. A fixed direction approach requires continuous thrust and is less efficient than other approach strategies. For this reason this kind of guidance law is only used at the very end of the rendezvous phase when the vehicles are in close proximities and the linearization assumptions provide a very good approximation.

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