Adaptable Recursive Update Filter

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I. Introduction

The Kalman filter [1, 2] is an optimal linear estimator for linear dynamic systems subject to linear measurements. The extended Kalman filter [3] (EKF) is a nonlinear approximation of the Kalman filter which assumes small estimation errors and approximates them to first order to calculate their covariance matrix. The EKF is a linear estimator that relies on the additional assumption that the first order approximation is valid. Nonlinear filters with a polynomial update up to arbitrary order are known [4]; knowledge of moments of the conditional estimation error distribution higher than the second are needed for these updates. Techniques exist to overcome some of the limitations of the EKF linearization assumption. The Gaussian second order filter (GSOF) [7] takes into account second-order terms assuming the prior error distribution is Gaussian. The iterated extended Kalman filter (IEKF) [3] recursively improves the center of the Taylor series expansion for a better linearization. The unscented Kalman filter [8] (UKF) is able to retain higher-order terms of the Taylor series expansion. Underweighting [9] is an *ad hoc* technique to compensate for the second order effects without actually computing them. The Recursive Update Filter (RUF) [10] applies the update gradually and re-linearizes at each recursion, hence avoiding linearization problems and providing a consistent covariance.

The number of recursions of the RUF is a user-defined parameter that needs to be selected by addressing two conflicting design objectives. On the one hand, the number of iterations should be chosen high to improve performance; the more recursions the more often the algorithm re-linearizes and the better the non-linearity of the measurement is followed. On the other hand, whenever

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computational time is of concern, it is desirable to reduce the number of recursions. In general, the higher the degree of nonlinearity of the measurement, the more recursions are needed. A good indicator of the performance of the algorithm is the post-update residual (actual measurement minus estimated measurement computed with the updated state). The residual should be consistent with its predicted covariance. Discrepancies between the two indicate the nonlinear effects are of concern and more iterations are needed.

The estimate $\hat{\mathbf{x}}$ provided by the IEKF is equivalent to using a Gauss-Newton method [11] to minimize the following nonlinear least-squares problem:

$$\min_{\mathbf{x}} \mathcal{J} = (\mathbf{x} - \hat{\mathbf{x}}^-)^{\mathrm{T}} (\mathbf{P}^-)^{-1} (\mathbf{x} - \hat{\mathbf{x}}^-) + (\mathbf{h}(\mathbf{x}) - \mathbf{y})^{\mathrm{T}} \mathbf{R}^{-1} (\mathbf{h}(\mathbf{x}) - \mathbf{y}). \tag{1}$$

Newton methods converge on a given interval when inside the interval the derivative of the function is non-zero and the higher order effects are not dominating. When the higher order effects are significant, the method can overshoot and diverge. Much like the IEKF, the RUF fails when the derivative is zero (as does the EKF). However, because the update is applied gradually, no overshooting occurs. While the IEKF is a Gauss-Newton method, the RUF closely resembles a gradient method. Line searching techniques [12] can avoid divergence of Newton methods, they usually rely on numerical optimization and their complexity is not usually suitable for real-time applications. Standard numerical optimization techniques can be used to calculate the number of iterations or recursions, but they require repeated calculations of the performance index Eq. (1), which requires the inversion of the filter covariance matrix, which can prove computationally expensive.

This paper documents how to adaptably and autonomously select the number of recursions. The algorithm takes advantage of the aforementioned relation between the measurement residuals and their covariance. Only the covariance of the residual needs to be inverted, which is usually of a much smaller dimension than the state covariance.

II. The Recursive Update Filter

This section presents the recursive update filter as first derived in [10]. The Kalman filter equations with correlated measurement and process noise [13] are needed since the recursive update methodology has correlated state vector and measurement error for every recursion except the

first. The cross-covariance at time t_k between the true (unknown) state $\mathbf{x}_k$ and the zero-mean measurement noise $\boldsymbol{\eta}_k$ is

$$\mathbf{C}_k = \mathrm{E}\left\{(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\boldsymbol{\eta}_k^{\mathrm{T}}\right\}, \tag{2}$$

where $\hat{\mathbf{x}}_k^-$ is the prior estimated state. A linear measurement $\mathbf{y}_k$ is first assumed

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \boldsymbol{\eta}_k. \tag{3}$$

The linear unbiased update is given by

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-). \tag{4}$$

The optimal gain in terms of minimum variance estimation is

$$\mathbf{K}_k = \left(\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k\right)\mathbf{W}_k^{-1} \tag{5}$$

where $\mathbf{P}_k^-$ is the prior estimation error covariance matrix, $\mathbf{W}_k$ is the residual covariance matrix, and $\mathbf{R}_k$ is the measurement error covariance matrix. It is assumed throughout this work that all errors and noises are zero mean. The updated estimation error covariance is given by

$$\begin{aligned}\mathbf{P}_k^+ =&(\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^-(\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^{\mathrm{T}} - \\ &(\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)\mathbf{C}_k\mathbf{K}_k^{\mathrm{T}} - \mathbf{K}_k\mathbf{C}_k^{\mathrm{T}}(\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)^{\mathrm{T}},\end{aligned} \tag{6}$$

where $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix, n being the size of the state vector. Finally, the updated cross-covariance is

$$\mathbf{C}_k^+ =(\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)\mathbf{C}_k - \mathbf{K}_k\mathbf{R}_k. \tag{7}$$

If the same measurement is processed multiple times, all recursions except the first have a zero optimal gain. This result is to be expected; after the first update all the information from the measurement is extracted, therefore processing the same measurement again, no additional update should occur. The RUF algorithm applies only a fraction of the optimal update during the first recursion, hence the subsequent optimal gain is not zero. The RUF gain is given by:

$$\mathbf{K}_k = \gamma_k^{(i)}\left(\mathbf{P}_k^{(i-1)}(\mathbf{H}_k^{(i-1)})^{\mathrm{T}} + \mathbf{C}_k^{(i-1)}\right)(\mathbf{W}_k^{(i-1)})^{-1}, \tag{8}$$

where i is the current recursion number and $\gamma_k^{(i)} \in (0, 1]$ determines which fraction of the optimal update is performed. With two recursions, after the first iteration only half of the optimal update is applied ($\gamma_k^{(1)} = 0.5$). During the second iteration the full optimal gain is applied ($\gamma_k^{(2)} = 1$) such that the net result after both updates is identical to the standard Kalman algorithm. If three iterations are performed, and each iteration updates one third of the total, the first gain is scaled by $\gamma_k^{(1)} = 1/3$. The second coefficient is $\gamma_k^{(2)} = 1/2$ because the remaining optimal update is two thirds of the total, the last coefficient is $\gamma_k^{(3)} = 1$. This procedure can be expanded to an arbitrary number N of iterations.

For the linear measurement case RUF is equivalent to the Kalman filter, making the iterations redundant. The benefits of this approach lie in the nonlinear measurement case when nonlinear effects can produce an inconsistent covariance and even cause divergence. The nonlinear measurement is given by

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \boldsymbol{\eta}_k \tag{9}$$

the algorithm for the nonlinear recursive update is given by Table 1.

Table 1 Recursive update, nonlinear measurements

Initialize the cross covariance $\mathbf{C}_k$ to zero
Repeat the following recursions N times:
Calculate the measurement Jacobian $\mathbf{H}_k$ evaluated at the current estimated state
Calculate the current recursion update fraction: $\gamma_k^{(i)} = 1/(N + 1 - i)$
Calculate the Kalman gain with Eq. (8)
Calculate the updated estimated state with Eq. (4)
Calculate the updated covariance with Eq. (6)
Calculate the updated cross-covariance with Eq. (7)

The choice of $\gamma_k^{(i)}$ is such that, in the linear case, all the steps have the same length, which is reasonable when the number of recursions N is fixed and determined *a priori* . However, notice that there is no need for $\gamma_k^{(i)}$ to be chosen as indicated in Table 1. The sequence $\gamma_k^{(i)}$ only has two requirements: it must be composed by numbers between zero and one (extremes excluded) for all elements except the last which must be equal to one. Consider the linear case once again and the

following coefficients:

$$\gamma_k^{(i)} = \begin{bmatrix} 1/2 & 1/4 & 1 \end{bmatrix} \tag{10}$$

with this choice the first recursion performs half of the update, the second recursion performs one fourth of the other half, or one eighth of the total. The last recursion performs all is left, or three eighths of the total. Since a linear measurement is assumed, these recursions are equivalent to a single update with the Kalman filter and to three recursive updates with coefficients $\begin{bmatrix} 1/3 & 1/2 & 1 \end{bmatrix}$.

III. Autonomous Selection of Iterations

As mentioned before, the IEKF is the Gauss-Newton solution to the nonlinear least squares problem given by Eq. (1). In the linear measurement case the performance index becomes:

$$\min_{\mathbf{x}_k} \mathcal{J} = (\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^{\mathrm{T}} (\mathbf{P}_k^-)^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_k^-) + (\mathbf{H}_k \mathbf{x}_k - \mathbf{y}_k)^{\mathrm{T}} \mathbf{R}_k^{-1} (\mathbf{H}_k \mathbf{x}_k - \mathbf{y}_k). \tag{11}$$

Similarly to doing a line search in the IEKF, it is possible is to select the number of steps such that each of them guarantees the reduction of the performance index. There are two issues with guaranteeing descent of Eq. (1). The recursive update filter always starts each iteration from the previous step; hence, in order to recalculate the performance index it is necessary to invert $\mathbf{P}^{(i)}$ multiple times, which is often a matrix of large dimensions and usually much larger than the measurement noise matrix or the residual covariance matrix. The second reason is that while choosing an updated state that guarantees that the performance index decreases results in a better state estimate, this approach does not guarantee that the state update occurs inside a region where the linearization assumption is valid. The goal of this work is to choose the RUF step size such that each small update occurs in the linear region in order to provide a consistent covariance. The performance index evaluated at the prior state is given by

$$\mathcal{J}(\hat{\mathbf{x}}_k^-) = (\mathbf{H}_k \hat{\mathbf{x}}_k^- - \mathbf{y}_k)^{\mathrm{T}} \mathbf{R}_k^{-1} (\mathbf{H}_k \hat{\mathbf{x}}_k^- - \mathbf{y}_k) = \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{R}_k^{-1} \boldsymbol{\epsilon}_k, \tag{12}$$

where $\boldsymbol{\epsilon}_k = \mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-$ is the measurement residual. Evaluating the performance index at the posterior state estimate

$$\begin{aligned}\mathcal{J}(\hat{\mathbf{x}}_k^+) &= (\hat{\mathbf{x}}_k^+ - \hat{\mathbf{x}}_k^-)^{\mathrm{T}}(\mathbf{P}_k^-)^{-1}(\hat{\mathbf{x}}_k^+ - \hat{\mathbf{x}}_k^-) + (\mathbf{H}_k\hat{\mathbf{x}}_k^+ - \mathbf{y}_k)^{\mathrm{T}}\mathbf{R}_k^{-1}(\mathbf{H}_k\hat{\mathbf{x}}_k^+ - \mathbf{y}_k) \\ &= \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}}(\mathbf{P}_k^-)^{-1}\mathbf{K}_k\boldsymbol{\epsilon}_k + (\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}\mathbf{R}_k^{-1}\boldsymbol{\epsilon}_k^+ \end{aligned} \tag{13}$$

where $\boldsymbol{\epsilon}_k^+$ the post-fit residual. In the linear case $\mathcal{J}(\hat{\mathbf{x}}^-) \geq \mathcal{J}(\hat{\mathbf{x}}^+)$ and $\boldsymbol{\epsilon}_k^+$ is given by

$$\boldsymbol{\epsilon}_k^+ = \mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^+ = \mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^- - \mathbf{H}_k\mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-) = (\mathbf{I} - \mathbf{H}_k\mathbf{K}_k)\boldsymbol{\epsilon}_k \tag{14}$$

$$= \left(\mathbf{I} - \mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\right)\boldsymbol{\epsilon}_k \tag{15}$$

$$= \mathbf{R}_k(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\boldsymbol{\epsilon}_k = \mathbf{R}_k\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k \tag{16}$$

where $\mathbf{W}_k = \mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k$; therefore the covariance of $\boldsymbol{\epsilon}_k^+$ is given by

$$\mathbf{W}_k^+ = \mathbf{R}_k\mathbf{W}_k^{-1}\mathbf{R}_k = (\mathbf{I} - \mathbf{H}_k\mathbf{K}_k)\mathbf{W}_k(\mathbf{I} - \mathbf{H}_k\mathbf{K}_k)^{\mathrm{T}} \tag{17}$$

and it follows that

$$(\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}(\mathbf{W}_k^+)^{-1}\boldsymbol{\epsilon}_k^+ = \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k. \tag{18}$$

In the scalar measurement case this means that the residual normalized by its standard deviation is the same before and after the update. The posterior performance index is given by

$$\begin{aligned}\mathcal{J}(\hat{\mathbf{x}}_k^+) &= \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}}(\mathbf{P}_k^-)^{-1}\mathbf{K}_k\boldsymbol{\epsilon}_k + (\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}\mathbf{R}_k^{-1}\boldsymbol{\epsilon}_k^+ \\ &= \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\mathbf{H}_k\mathbf{P}_k^-(\mathbf{P}_k^-)^{-1}\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k + \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\mathbf{R}_k\mathbf{R}_k^{-1}\mathbf{R}_k\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k \\ &= \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k + \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\mathbf{R}_k\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k \\ &= \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k = (\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}(\mathbf{W}_k^+)^{-1}\boldsymbol{\epsilon}_k^+ \leq \mathcal{J}(\hat{\mathbf{x}}_k^-) = \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{R}_k^{-1}\boldsymbol{\epsilon}_k \end{aligned} \tag{19}$$

Notice that the last inequality holds because

$$\mathbf{R}_k \leq \mathbf{W}_k = \mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k. \tag{20}$$

The same equality relation between normalized prior and posterior residual ratios is also valid

for the suboptimal correlated linear update:

$$\boldsymbol{\epsilon}_k^+ = \mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^+ = \mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^- - \mathbf{H}_k\mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-) = (\mathbf{I} - \mathbf{H}_k\mathbf{K}_k)\boldsymbol{\epsilon}_k \tag{21}$$

$$= \Big(\mathbf{I} - \gamma\mathbf{H}_k(\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k)\left(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{H}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}\right)^{-1}\Big)\boldsymbol{\epsilon}_k \tag{22}$$

$$= \Big((1-\gamma)\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + (1-\gamma)\mathbf{H}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}\Big)\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k = \mathbf{A}_k\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k \tag{23}$$

where the residual covariance is now $\mathbf{W}_k = \mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{H}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}$, once again $\mathbf{C}_k$ is the correlation between prior state estimation error and measurement noise, and γ is a coefficient between zero and one specifying the lengths of the update. The covariance of $\boldsymbol{\epsilon}_k^+$ is given by

$$\mathbf{W}_k^+ = \mathbf{A}_k\mathbf{W}_k^{-1}\mathbf{A}_k^{\mathrm{T}} \tag{24}$$

and once again $(\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}(\mathbf{W}_k^+)^{-1}\boldsymbol{\epsilon}_k^+ = \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k$. Notice that $\mathbf{W}_k^+$ can be calculated as

$$\mathbf{W}_k^+ = \mathbf{H}_k\mathbf{P}_k^+\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{H}_k\mathbf{C}_k^+ + (\mathbf{C}_k^+)^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}, \tag{25}$$

where $\mathbf{C}_k^+$ is given by Eq. (7) and, in the nonlinear case, the measurement Jacobian $\mathbf{H}_k$ is evaluated at the updated state estimate.

In the EKF case

$$\boldsymbol{\epsilon}_k^+ = \mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^+) = \mathbf{y}_k - \Big(\mathbf{h}(\hat{\mathbf{x}}_k^-) + \mathbf{H}_k\mathbf{K}_k(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)) + \boldsymbol{\delta}\Big)$$

$$= (\mathbf{I} - \mathbf{H}_k\mathbf{K}_k)(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)) + \boldsymbol{\delta} = (\mathbf{I} - \mathbf{H}_k\mathbf{K}_k)\boldsymbol{\epsilon}_k + \boldsymbol{\delta} = \mathbf{R}_k\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k + \boldsymbol{\delta} \tag{26}$$

where $\boldsymbol{\delta}$ is a vector containing the higher order terms of the Taylor series expansion of the measurement function. It follows that

$$(\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}(\mathbf{W}_k^+)^{-1}\boldsymbol{\epsilon}_k^+ = \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k + \kappa \tag{27}$$

where κ is some scalar function of the higher order terms. In the nonlinear case $(\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}(\mathbf{W}_k^+)^{-1}\boldsymbol{\epsilon}_k^+ \simeq \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k$ only when the higher order effects of the measurement nonlinearity are negligible. This property can be used to select the length of of the update step; the step can be decreased until the two quantities match within a desired tolerance. By selecting the proper tolerance in enforcing $(\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}(\mathbf{W}_k^+)^{-1}\boldsymbol{\epsilon}_k^+ \simeq \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k$, it is possible to ensure that the updated state produces a smaller performance index. The algorithm to adaptably determine the number of recursions N is summarized in Table 2.

Table 2 Autonomous selection of N, Recursive Update Filter

Initialize the cross covariance $\mathbf{C}_k$ to zero and N to one
Repeat the following recursions N times:
Calculate the measurement Jacobian: $\mathbf{H}_k$ evaluated at the current estimated state
Calculate the current recursion update fraction: $\gamma_k^{(i)} = 1/(N+1-i)$
Calculate the Kalman gain with Eq. (8)
Calculate the updated estimated state with Eq. (4)
Calculate the updated covariance with Eq. (6)
Calculate the updated cross-covariance with Eq. (7)
Caluculate the updated residual and its covariance (with an updated measurement Jacobian)
If the posterior residual ratio does not match the prior ratio within a threshold ϑ increase N by one and repeat the current recursion

A few comments are in order. In Table 2 the positive scalar ϑ is a design variable indicating the amount of nonlinear effects tolerated while selecting N. A value of zero means $(\boldsymbol{\epsilon}_k^+)^{\mathrm{T}}(\mathbf{W}_k^+)^{-1}\boldsymbol{\epsilon}_k^+ = \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{W}_k^{-1}\boldsymbol{\epsilon}_k$ is strictly enforced. Selecting $\vartheta = 0$ is a poor choice for nonlinear systems as the algorithm will never reach its exit condition. The algorithm selects the number of recursions based on the length of the first recursion. All of the successive recursions check that the current length is appropriate, if not the length is decreased. Assume for example that during the first recursion $N = 2$ is selected. The second recursion ($i = 2$) starts with $N = 2$ and calculates $\gamma_k^{(2)} = 1$, if the normalized residuals check fails the total number of recursions is increased to $N = 3$, which makes $\gamma_k^{(2)} = 0.5$. A third recursion is therefore called with $\gamma_k^{(3)} = 1$.

Notice that the residual covariance matrix is assumed invertible. This condition is always satisfied so long as $\mathbf{R}_k$ is invertible.

IV. Numerical Results

This section introduces an example of great interested to aerospace GN&C, a spacecraft rendezvous problem. It is well known that spacecraft rendezvous are subject to failure of the linearization assumptions when very precise nonlinear measurements become first available [14]. This issue is generally resolved with underweighting [9], a technique first developed and used by the Space

Shuttle. The relative trajectory is shown in Fig. 1. The chaser spacecraft starts one hundred meters in front of the target and performs a +V-bar approach (i.e., it approaches the target along its positive inertial velocity direction). The estimated state $\hat{\mathbf{x}}$ contains relative position and velocity, the initial filter covariance $\mathbf{P}_0$ is obtained setting the state uncertainty to 10 m in position and 0.05 m/s in velocity, per axis. The dynamics is governed by the Clohessy–Wiltshire equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \boldsymbol{\nu} \qquad \mathbf{P}_0 = \begin{bmatrix} 10^2\mathbf{I}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & 0.05^2\mathbf{I}_{3\times3} \end{bmatrix} \tag{28}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{I}_{3\times3} \\ \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} \tag{29}$$

and

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -n^2 & 0 \\ 0 & 0 & 3n^2 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 2n \\ 0 & 0 & 0 \\ -2n & 0 & 0 \end{bmatrix},$$

with $n = 0.0011$ rad/s is the target's orbital rate. The process noise is zero mean with spectral density

$$\mathbf{Q} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & 10^{-9}\mathbf{I}_{3\times3} \end{bmatrix} \tag{30}$$

A range measurement ρ with 0.1 m accuracy and two bearing angles with 0.1 deg accuracy are available every two seconds. The bearing angles are azimuth $\alpha = \tan^{-1}(\mathbf{x}(1)/\mathbf{x}(2))$ and elevation $\epsilon = \sin^{-1}(\mathbf{x}(3)/\rho)$.

$$\mathbf{y} = \begin{bmatrix} \rho \\ \alpha \\ \epsilon \end{bmatrix} + \boldsymbol{\eta} \tag{31}$$

$$\mathrm{E}\{\boldsymbol{\eta}\} = \mathbf{0}, \qquad \mathrm{E}\left\{\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}}\right\} = \begin{bmatrix} 0.1^2 & \mathbf{O}_{1\times2} \\ \mathbf{O}_{2\times1} & (0.1\pi/180)^2\mathbf{I}_{2\times2} \end{bmatrix} \tag{32}$$

One hundred runs are performed in which the initial estimation error and the sensor errors are dispersed using zero mean Gaussian independent random variables. Fig. 2 shows the performance of

the EKF, the lighter lines are the 100 samples of the estimation error, while the thicker black lines are the EKF 3σ predicted standard deviations (three times the square roots of the diagonals elements of the EKF's covariance matrix). The EKF is overly optimistic in predicting its performance. Eventually however the EKF is able to recover. Under similar but more severe circumstances, it is also possible for the EKF to diverge [9].

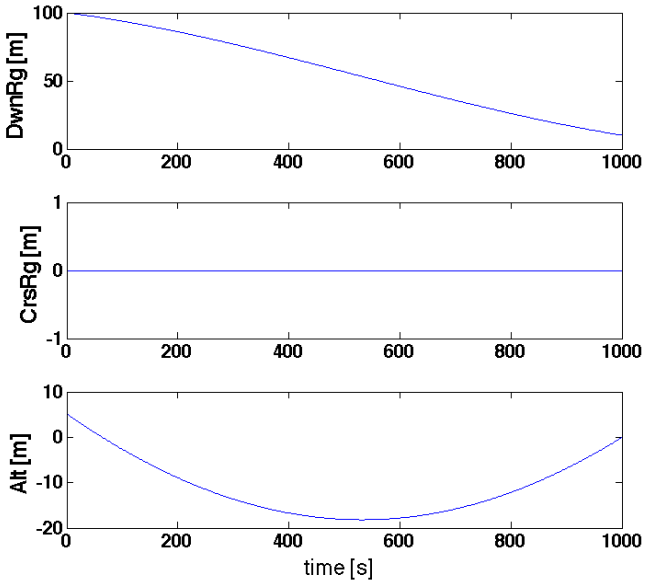


Fig. 1 Trajectory.

Fig. 3 shows the performance of the proposed algorithm. It can be seen that the filter outperforms the EKF during the first few hundred seconds. The recursive update filter is also able to predict its performance as the sample covariance of the actual errors from the Monte Carlo analysis matches the algorithm's prediction. In Fig. 3 most error samples are within the 1σ prediction and all of them are within the 3σ values. The performance of the two algorithms in estimating velocity is very similar to that of the position shown in Figs. 2 and 3. The algorithm selects the number of recursions to be equal to one for at all times except the very first one. The reason is that the algorithm is able to extract all the information out of the nonlinear measurement, therefore the uncertainty collapses after incorporating the first measurement and in all subsequent measurement updates the nonlinear effects are negligible. Underweighting and the Gaussian second order filter are other methodologies to achieve covariance consistency in this situation. However, they are not

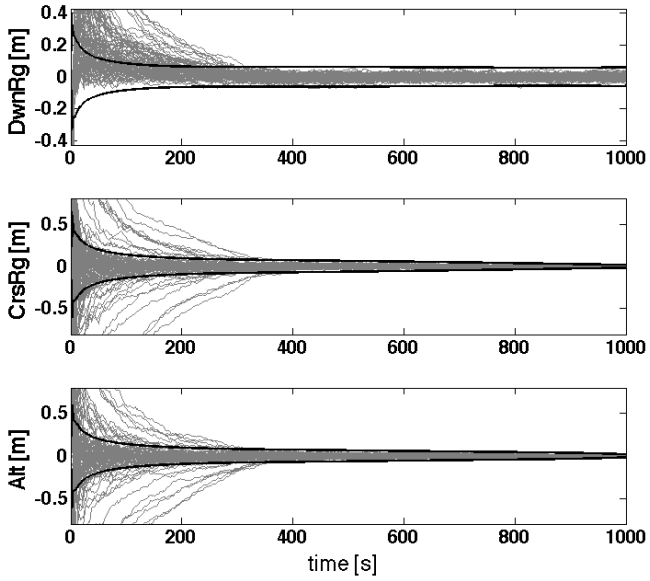


Fig. 2 EKF Estimation Error and 3σ Predicted Standard Deviation.

able to extract all the information out of the nonlinear measurement. Rather, they intentionally de-weight the measurement contributions in order to compensate for the nonlinear effects, hence these two algorithms could have significant nonlinear effects for measurements other than the first. If a long measurement blackout occurs, or if another much more precise sensor becomes available, the nonlinear effects can be significant again; the proposed algorithm would recognize this fact and select the number of recursions appropriately. This scenario could occur, for example, when transitioning from a long-range relative sensor such as a radar to a lidar that is much more precise.

In this numerical example, the number of recursions at the first step varies between 3 and 10 depending on the particular run, as seen in Fig. 4. Fig. 5 shows the distribution of the number of recursions selected by the algorithm when processing the first measurement.

V. Conclusions

Nonlinear estimators such as the iterated extended Kalman filter or the recursive update filter (RUF) outperform the extended Kalman filter in the presence of nonlinear measurements. This paper documents how to adaptably and autonomously select the number of recursions performed by RUF. The algorithm takes advantage of the relation between the measurement residuals and

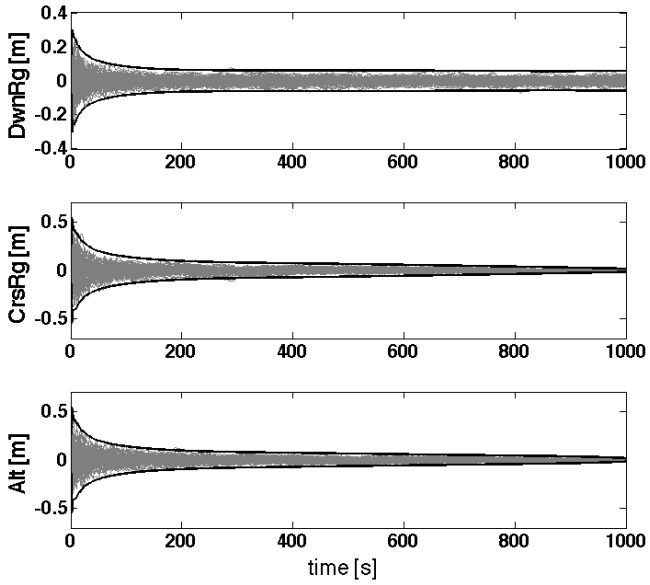
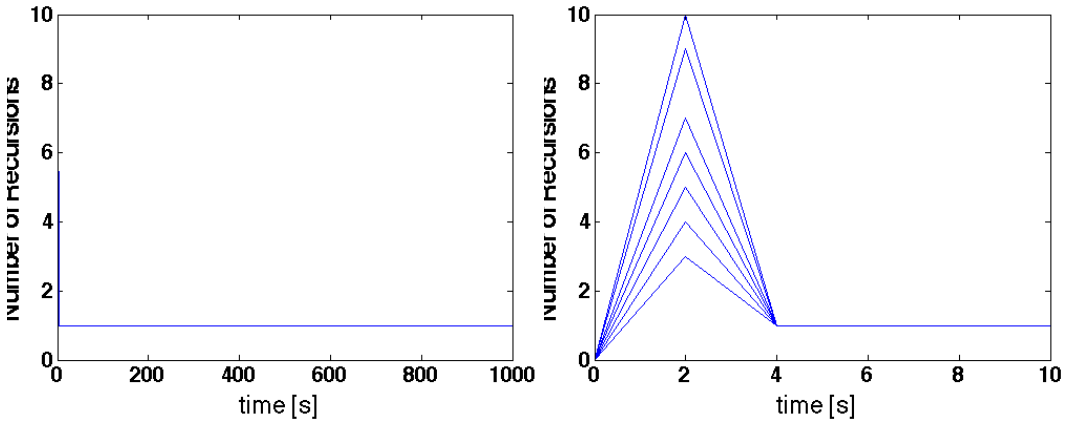


Fig. 3 Recursive Update Filter Estimation Error and 3σ Predicted Standard Deviation



(a) Zoom-Out View

(b) Zoom-In View

Fig. 4 Number of Recursions Selected by Algorithm

their covariance to keep each recursion in an area where the linearization assumption holds. In the presence of significant measurement nonlinearities, the proposed technique improves covariance consistency with respect to the extended Kalman filter. The algorithm extracts all the information out of a nonlinear measurement by breaking a large update into multiple smaller updates. Typically after the very first measurement is incorporated the subsequent updates are small, and the algorithm is capable of recognizing this fact and choosing a single recursion, reducing to the extended Kalman

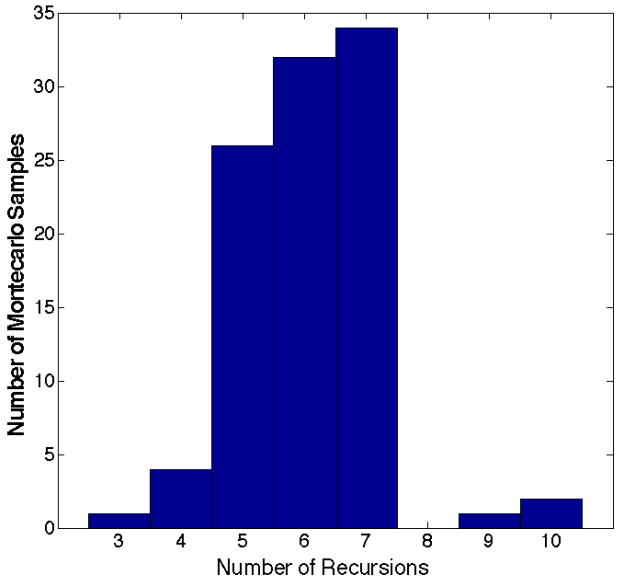


Fig. 5 100 Monte Carlo Histogram of Recursions During the First Measurement Update

filter. If at any subsequent time measurement nonlinearities become significant again, the algorithm re-introduces multiple recursions to mitigate them. This scenario could occur, for example, after a long measurement black-out or when another, more precise, sensor becomes available. A spacecraft rendezvous example is provided to demonstrate the validity of the proposed approach.

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