Quaternion Estimation and Norm Constrained Kalman Filtering

Renato Zanetti* and Robert H. Bishop†

The University of Texas, Austin, Texas, 78712

An analysis and comparison of two different strategies to implement the quaternion Kalman filter is presented. Circumstances under which the two strategies are equivalent will be investigated, together with conditions for covariance singularity. Normalization will also be analyzed. Optimality will be shown in a stochastic sense and not only in a geometric sense as previously reported.

I. Introduction

Attitude estimation has been the topic of much research and debate in the past two decades.[1] The interest arises from the fact that the representation of the attitude is not a vector space and redundancy is necessary to avoid singularities and discontinuities.[2] For realtime space applications, the quaternion-of-rotation is a favorite attitude representation. In sequential realtime quaternion estimation two schools of thinking received the most attention: the Additive Extended Kalman Filter[3] (AEKF) and the Multiplicative Extended Kalman Filter[19] (MEKF). While the additive approach resembles closely the standard extended Kalman filter (EKF), several shortcomings of the AEKF were pointed out. These are

1. The estimation error does not have a physical meaning.
2. The estimation error covariance becomes ill-conditioned.
3. It requires a brute force normalization procedure.

Theoretical studies show that the covariance in the AEKF should be nearly singular,[4] while practical applications do not reveal the problem.[5] In this work, we provide arguments to show that in the presence of process noise there is no need for the covariance matrix to be ill-conditioned.

Both the AEKF and MEKF necessitate restoring the norm constraint after the update. The most obvious method to accomplish this is to scale the updated quaternion by its norm, thereby minimizing the Euclidean distance between the unconstrained and the constrained estimates.[6] The main focus of this work is to obtain the optimal estimate while simultaneously constraining the norm. The result is that the normalization process provides the unitary estimate with minimum mean square error–a fact heretofore unproven.

It is well-known that the Kalman filter provides the *unconstrained* optimal solution of the stochastic estimation problem.[7–9] The Kalman filter algorithm has two main phases: the state estimate propagation phase between measurements, and the state estimate update phase when measurements become available. Unconstrained implies that the optimal state estimate is not constrained during the state estimate update phase as the measurements are processed. The Kalman filter provides the optimal state estimate considering n degrees of freedom (that is, the entire vector space $\Re^n$). However, if r state constraints are applied, the degrees of freedom are reduced to $n-r$. Simply projecting the unconstrained solution into the constrained space will not guarantee optimality.

It is assumed throughout this paper that through the mathematical model (that is, through the state equation) the underlying physics, including the state constraints, are satisfied during periods between measurements. The mathematical model of the system should adequately represent any desired state constraints.

*PhD Graduate Student, Department of Aerospace Engineering and Engineering Mechanics, 210 East 24th Street, W. R. Woolrich Laboratories, 1 University Station.

†Professor and Chairman, Department of Aerospace Engineering and Engineering Mechanics, 210 East 24th Street, W. R. Woolrich Laboratories, 1 University Station, AIAA Associate Fellow.

The challenge is to modify the Kalman filter solution so as to constrain the state update appropriately, while retaining the fundamental properties and characteristics of the familiar discrete Kalman filer algorithm. In the discrete formulation of the Kalman filter, the state can be related to the control algebraically. We can then view the optimization problem as a parameter optimization problem, therefore the state constraint can be expressed as a control constraint.

One method of introducing state constraints is to use pseudo-measurements.[10] The fundamental idea is to introduce a perfect measurement (hence the use of the term "pseudo-measurement") consisting of the constraint equation into the estimation solution. This approach has two shortcomings. First, the use of a perfect measurement results in a singular estimation problem known to occur when processing noise-free measurements in a Kalman filter. A small noise can be added to the pseudo-measurement to address the singularity; however with the noise introduced, the constraint is no longer exactly satisfied. Second, when the constraint is nonlinear (as is the case in this paper), after the linearization of the measurement equation consistent with the EKF algorithm, the constraint is no longer satisfied exactly.

One can consider state constraints when considering the optimization problems based on least squares methods. The solution to the least squares problem in the presence of linear equality constraints is found in Lawson.[11] Another approach is to project the Kalman solution into the desired subspace. Since the projection can be done in different ways, a performance index can be defined to find the optimal projection. The optimal projection for the linear state equality constraint problem is presented in Simon and Chia.[12] The projection of the Kalman solution can be done at any time, not only during the update.

II. Nomenclature

The quaternion format used here is vector first and scalar last, $\mathbf{Q} = \begin{bmatrix}\mathbf{q}^{\mathrm{T}} & q_4\end{bmatrix}^{\mathrm{T}}$. The rotation matrix from one frame to another (say from inertial to body) is given by:

$$\mathbf{T}(\mathbf{Q}) = \left(q_4^2 - \mathbf{q}^{\mathrm{T}}\mathbf{q}\right)\mathbf{I}_{3\times3} - 2q_4[\mathbf{q}\times] + 2\mathbf{q}\mathbf{q}^{\mathrm{T}}.$$

The vector product matrix $[\mathbf{q}\times]$ is defined such that

$$[\mathbf{q}\times]\mathbf{v} = \mathbf{q}\times\mathbf{v}.$$

The quaternion product $\otimes$ is defined such that the quaternions are multiplied in the same order as the attitude matrices. Originally, Hamilton defined the product in opposite order. Hamilton's product will be denoted by $\circledast$, and

$$\mathbf{Q}\otimes\mathbf{P} = \mathbf{P}\circledast\mathbf{Q},$$

where $\mathbf{Q}$ and $\mathbf{P}$ are quaternions. The quaternion product matrices are defined as follows

$$\mathbf{Q}\otimes\mathbf{P} = [\mathbf{Q}\otimes]\,\mathbf{P} = [\mathbf{P}\circledast]\,\mathbf{Q},$$

from which

$$[\mathbf{Q}\otimes] = [\mathbf{\Psi}(\mathbf{Q})\ \ \mathbf{Q}] \qquad \text{where} \qquad \mathbf{\Psi}(\mathbf{Q}) = \begin{bmatrix} q_4\mathbf{I}_{3\times3} - [\mathbf{q}\times] \\ -\mathbf{q}^{\mathrm{T}} \end{bmatrix}$$

and

$$[\mathbf{P}\circledast] = [\mathbf{\Xi}(\mathbf{P})\ \ \mathbf{P}] \qquad \text{where} \qquad \mathbf{\Xi}(\mathbf{P}) = \begin{bmatrix} p_4\mathbf{I}_{3\times3} + [\mathbf{p}\times] \\ -\mathbf{p}^{\mathrm{T}} \end{bmatrix}.$$

The kinematics of the quaternion are given by

$$\dot{\mathbf{Q}} = \frac{1}{2}\begin{bmatrix}\boldsymbol{\omega} \\ 0\end{bmatrix}\otimes\mathbf{Q} = \frac{1}{2}\mathbf{\Xi}(\mathbf{Q})\boldsymbol{\omega} = \frac{1}{2}\mathbf{\Omega}(\boldsymbol{\omega})\mathbf{Q} \qquad \text{where} \qquad \mathbf{\Omega}(\boldsymbol{\omega}) = \left[\begin{bmatrix}\boldsymbol{\omega} \\ 0\end{bmatrix}\circledast\right],$$

where the angular velocity $\boldsymbol{\omega}$ is expressed in the rotating frame (such as the body frame).

III. Mean-Square Error and Families of Estimators

In a stochastic setting, virtually every time a state estimate is computed, the estimate differs from the true state. The standard statistical measure to quantify the mismatch is the estimation error, defined as the difference between the true and the estimated states. When the true state and/or the measurements are random vectors, the expected value of the estimate error is of great interest. The mean-square error (MSE) of a scalar estimate $\widehat{x}$ is defined as[13]

$$E[e^2] = E[(\widehat{x} - x)^2].$$

For random vectors the MSE uses the vector 2-norm

$$E[\|\mathbf{e}\|^2] = E[\mathbf{e}^{\mathrm{T}}\mathbf{e}] = E[\operatorname{trace}(\mathbf{e}\mathbf{e}^{\mathrm{T}})] = \operatorname{trace} E[\mathbf{e}\mathbf{e}^{\mathrm{T}}].$$

The mean square error is a common way of evaluating estimators, and is equivalent to the trace of the mean-square estimate error covariance. It seems sensible to choose the estimator that on average has the smallest error (in a 2-norm sense). Unfortunately the set of all possible estimators is too big, and there is a lack of global optimality results. It is therefore common to look only at certain families of estimators, for which optimality results exist. One such a family are the unbiased estimators. An estimator is said to be unbiased when the estimation error is zero mean. This family of estimators is particularly important because it is desirable to have an estimator whose error is on average zero. However the best unbiased estimator could still perform worse (in a MSE sense) than other classes of estimators. The classical example is estimating the covariance of a normal variable where the maximum likelihood estimator has a lower MSE that the best unbiased estimator.[13]

The best unbiased estimator (in MSE sense) of $\mathbf{x}$ based on the observation $\mathbf{y}$ is given by:

$$\widehat{\mathbf{x}} = E[\mathbf{x}|\mathbf{y}].$$

Note that

$$E[\mathbf{e}\mathbf{e}^{\mathrm{T}}] = \operatorname{Cov}(\mathbf{e}, \mathbf{e}) + E[\mathbf{e}]\, E[\mathbf{e}^{\mathrm{T}}].$$

It follows that for *unbiased* estimators the MSE is given by

$$\operatorname{trace} \operatorname{Cov}(\mathbf{e}, \mathbf{e})$$

since the mean of the estimation error is zero. When searching for the best unbiased estimator, minimizing the trace of the estimation error covariance is equivalent to minimizing the MSE. This fact is generally not true, and it often makes little sense to minimize the covariance. Minimizing the covariance implies *shrinking* the estimation error around its mean, which is not necessarily desirable, since the mean might be large. Figure 1 illustrates this concept; the errors of two estimators are plotted. The first estimator is biased with $E[\mathbf{e}_1] = [5 \;\; 5]^{\mathrm{T}}$ and covariance $\operatorname{Cov}(\mathbf{e}_1, \mathbf{e}_1) = 0.04\, I_{2\times 2}$, while the second estimator is unbiased with covariance $\operatorname{Cov}(\mathbf{e}_2, \mathbf{e}_2) = I_{2\times 2}$. The biased estimator has clearly much smaller covariance but much larger MSE.

Other possible families of estimators for which optimality conditions can be satisfied are the linear and the affine estimators. The best linear estimator in MSE sense is

$$\widehat{\mathbf{x}} = E[\mathbf{x}\mathbf{y}^{\mathrm{T}}]E[\mathbf{y}\mathbf{y}^{\mathrm{T}}]^{-1}\mathbf{y}. \tag{1}$$

Notice that this estimator could be, and usually is, biased. The best affine estimator in MSE sense is

$$\widehat{\mathbf{x}} = \operatorname{Cov}(\mathbf{x}, \mathbf{y}) \operatorname{Cov}(\mathbf{y}, \mathbf{y})^{-1}(\mathbf{y} - E[\mathbf{y}]) + E[\mathbf{x}]. \tag{2}$$

The optimal affine estimator is unbiased. The best affine estimator is *an* unbiased estimator, and is not, generally, *the best* unbiased estimator. Of course in the presence of gaussian noise they are the same.

A. Recursive Estimation

It is not convenient to store all the measurements, rather it is desirable to process all measurement as they become available. Equations (1) and (2) are valid for every distribution of the measured random vector $\mathbf{y}$.

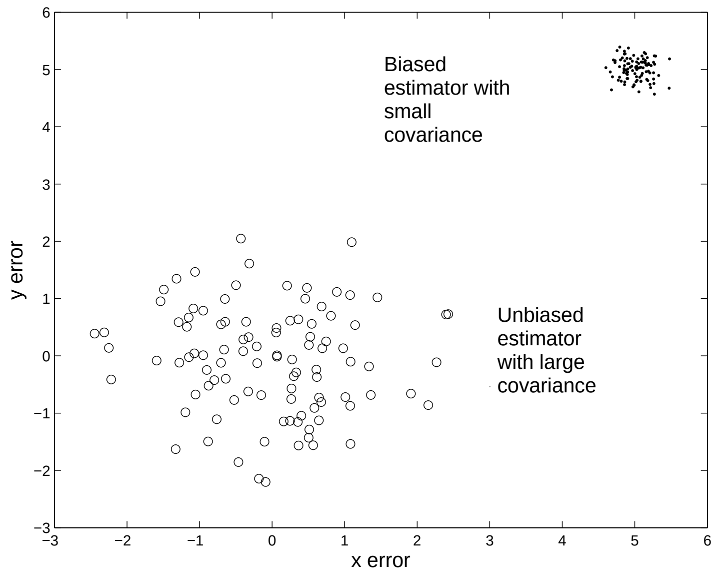

Figure 1. Comparison of mean square errors of two estimators.

In the attempt to derive a recursive estimator, we pay the price of having to assume a very special form for $\mathbf{y}$. Suppose that the measurement is given by

$$\mathbf{y}^{\mathrm{T}} = \begin{bmatrix} \mathbf{y}_1^{\mathrm{T}} & \mathbf{y}_2^{\mathrm{T}} \end{bmatrix},$$

and $\widehat{\mathbf{x}}_1$ is the best affine estimator based on $\mathbf{y}_1$. Define

$$\mathbf{P}_1 \triangleq E[(\mathbf{x} - \widehat{\mathbf{x}}_1)(\mathbf{x} - \widehat{\mathbf{x}}_1)^{\mathrm{T}}] = E[\mathbf{e}_1 \mathbf{e}_1^{\mathrm{T}}].$$

If

$$\mathbf{y}_2 = \mathbf{H}\mathbf{x} + \mathbf{u}, \qquad E[\mathbf{u}] = \mathbf{0}, \;\; E[\mathbf{u}\mathbf{u}^{\mathrm{T}}] = \mathbf{R},$$

and

$$E[\mathbf{x}\mathbf{u}^{\mathrm{T}}] = \mathbf{0}, \qquad E[\mathbf{y}_1 \mathbf{u}^{\mathrm{T}}] = \mathbf{0},$$

then the best affine estimator based on $\mathbf{y}$ is

$$\widehat{\mathbf{x}}_2 = \widehat{\mathbf{x}}_1 + \mathbf{P}_1 \mathbf{H}^{\mathrm{T}} (\mathbf{H}\mathbf{P}_1\mathbf{H}^{\mathrm{T}} + \mathbf{R})^{-1} (\widehat{\mathbf{y}}_2 - \mathbf{H}\widehat{\mathbf{x}}_1).$$

Since $\widehat{\mathbf{x}}_1$ is unbiased, $\mathbf{P}_1$ is the covariance of the estimation error. The covariance of the *a posteriori* estimation error is

$$\mathbf{P}_2 \triangleq E[(\mathbf{x} - \widehat{\mathbf{x}}_2)(\mathbf{x} - \widehat{\mathbf{x}}_2)^{\mathrm{T}}] = \mathbf{P}_1 - \mathbf{P}_1\mathbf{H}^{\mathrm{T}}[\mathbf{H}\mathbf{P}_1\mathbf{H}^{\mathrm{T}} + \mathbf{R}]^{-1}\mathbf{H}\mathbf{P}_1.$$

Under the same hypothesis, if $\widehat{\mathbf{x}}_1$ is the best linear estimator based on $\mathbf{y}_1$, then $\widehat{\mathbf{x}}_2$ is the best linear estimator based on $\mathbf{y}$.[14] Notice that in this case $\mathbf{P}_1$ and $\mathbf{P}_2$ represent mean-square errors and not error covariances.

In engineering applications emphasis is placed on the error covariance and how it should be minimized. It is therefore important to notice that minimizing the trace of the covariance derives from the desire to minimize the MSE of an unbiased estimator. When the mean of the estimation error is different from zero matrices $\mathbf{P}$ of the Kalman filter are mean-squares and contain the mean of the estimation error. For example, the quaternion estimation error

$$\mathbf{e}_Q \triangleq \mathbf{Q} - \widehat{\mathbf{Q}}$$

being zero-mean before the normalization process, does not guarantee that it will be zero-mean after brute force normalization.

The estimation error is a way of quantifying the difference between the true state and the estimated state, therefore it does not necessarily have to be physically meaningful. The same holds true for the MSE, which is one measure of the performance of the estimator. A small MSE it is desirable independent of the physics of the problem and independent of the covariance matrix (which might be singular). Like any performance index, its use could be replaced by another measure leading to a different optimization solution.

IV. Quaternion Estimation

Quaternion estimation has the goal of making the *distance* between the estimated and the estimate variables small. It is common to define this *distance* to be a quaternion-of-rotation itself, where

$$\delta\mathbf{Q} = \mathbf{Q} \otimes \widehat{\mathbf{Q}}^{-1}.$$

This approach guarantees a physical interpretation of the estimation error. The relation between the multiplicative error and the additive error is given by

$$\mathbf{e}_Q = \mathbf{Q} - \widehat{\mathbf{Q}} = \mathbf{Q} - \delta\mathbf{Q}^{-1} \otimes \mathbf{Q} = \mathbf{Q} - \begin{bmatrix} -\Xi(\mathbf{Q}) & \mathbf{Q} \end{bmatrix} \delta\mathbf{Q}.$$

The true state, $\mathbf{Q}$, is often treated as deterministic, and therefore taken outside the expectation operation. This operation is legitimate only in the absence of process noise. In the presence of process noise, $\mathbf{Q}$ is a random quantity and taking it outside the expected value results in an approximation, therefore only approximate conclusions can be inferred from this procedure.

The *covariance* which is associated with the additive Kalman filter is

$$\mathbf{P}_a = E[\mathbf{e}_Q \, \mathbf{e}_Q^{\mathrm{T}}]$$

and for the multiplicative approach we have

$$\mathbf{P}_{4m} = E[\delta\mathbf{Q} \, \delta\mathbf{Q}^{\mathrm{T}}].$$

In calculating $\mathbf{P}_a$, the Kalman filter does not compensate for the mean of $\mathbf{e}_Q$, therefore any theoretical study of $\mathbf{P}_a$ should not contain $E[\mathbf{e}_Q]$ unless it is proven to be zero. It was shown[15] that when the estimation error is a small rotation, $\mathbf{P}_{4m}$ becomes ill-conditioned. Assuming the true quaternion $\mathbf{Q}$ is deterministic and relating the two matrices, will result in $\mathbf{P}_a$ being ill-conditioned as well. However, when $\mathbf{Q}$ is a random vector it cannot be taken outside the expected value, and no conclusions can be made on $\mathbf{P}_a$ from the condition of $\mathbf{P}_{4m}$. The matrix $\mathbf{P}_a$ will depend on the joint distribution of $\mathbf{Q}$ and $\delta\mathbf{Q}$ and not solely on $\delta\mathbf{Q}$.

In the absence of process noise, the Kalman filter covariance will eventually converge to zero. Therefore, the fact that in the absence of process noise $\mathbf{P}_a$ becomes ill-conditioned is an expected characteristic of the AEKF scheme.

A. Relationship Between Additive and Multiplicative Error Representation

The MEKF is equivalent to the AEKF*. Let the measurement $\mathbf{y}$ be related to the quaternion through a nonlinear function $\mathbf{h}$ and noise $\mathbf{u}$ as

$$\mathbf{y} = \mathbf{h}(\mathbf{Q}) + \mathbf{u} = \mathbf{h}(\widehat{\mathbf{Q}} + \mathbf{e}_Q) + \mathbf{u} = \mathbf{h}(\delta\mathbf{Q} \otimes \widehat{\mathbf{Q}}) + \mathbf{u} = \mathbf{h}\left(\begin{bmatrix} \widehat{\mathbf{Q}} \circledast \end{bmatrix} \delta\mathbf{Q}\right) + \mathbf{u} \simeq \mathbf{h}\left(\Xi(\widehat{\mathbf{Q}})\delta\mathbf{q} + \widehat{\mathbf{Q}}\right) + \mathbf{u}$$

which is approximately

$$\mathbf{h}(\widehat{\mathbf{Q}} + \mathbf{e}_Q) \simeq \mathbf{h}(\widehat{\mathbf{Q}}) + \frac{d}{d\mathbf{e}_Q}\mathbf{h}(\widehat{\mathbf{Q}} + \mathbf{e}_Q)\bigg|_{\mathbf{e}_Q = \mathbf{0}} \mathbf{e}_Q, \text{ and}$$

$$\mathbf{h}\left(\Xi(\widehat{\mathbf{Q}})\delta\mathbf{q} + \widehat{\mathbf{Q}}\right) \simeq \mathbf{h}(\widehat{\mathbf{Q}}) + \frac{d}{d\delta\mathbf{q}}\mathbf{h}\left(\Xi(\widehat{\mathbf{Q}})\delta\mathbf{q} + \widehat{\mathbf{Q}}\right)\bigg|_{\delta\mathbf{q} = \mathbf{0}} \delta\mathbf{q}.$$

*Shuster[16] cites Ferraresi[17] to prove the equivalency, Ferraresi's work was not available to the authors.

Defining $\mathbf{H}(\mathbf{Q})$ as the jacobian of $\mathbf{h}(\mathbf{Q})$ and using the chain rule it follows that

$$\mathbf{H}_a(\widehat{\mathbf{Q}}) \triangleq \frac{d}{d\mathbf{e}_Q}\mathbf{h}(\widehat{\mathbf{Q}}+\mathbf{e}_Q)\Big|_{\mathbf{e}_Q} = \mathbf{H}(\widehat{\mathbf{Q}}), \text{ and}$$
$$\mathbf{H}_m(\widehat{\mathbf{Q}}) \triangleq \frac{d}{d\delta\mathbf{q}}\mathbf{h}\left(\boldsymbol{\Xi}(\widehat{\mathbf{Q}})\delta\mathbf{q}+\widehat{\mathbf{Q}}\right)\Big|_{\delta\mathbf{q}=\mathbf{0}} = \mathbf{H}(\widehat{\mathbf{Q}})\boldsymbol{\Xi}(\widehat{\mathbf{Q}}) = \mathbf{H}_a(\widehat{\mathbf{Q}})\boldsymbol{\Xi}(\widehat{\mathbf{Q}}).$$

Let $\mathbf{P}_m \in \Re^{3\times 3}$ be the multiplicative error covariance matrix. If the *a priori* estimates are the same and if the *a priori* covariances obey the following relation

$$\mathbf{P}_a^- = \boldsymbol{\Xi}(\widehat{\mathbf{Q}}^-)\mathbf{P}_m^-\boldsymbol{\Xi}(\widehat{\mathbf{Q}}^-)^{\mathrm{T}},$$

then the additive Kalman gain is (dropping the arguments of the matrix functions)

$$\begin{aligned}\mathbf{K}_a &= \mathbf{P}_a^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{P}_a^-\mathbf{H}^{\mathrm{T}}+\mathbf{R})^{-1} = \boldsymbol{\Xi}\mathbf{P}_m^-\boldsymbol{\Xi}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}(\mathbf{H}\boldsymbol{\Xi}\mathbf{P}_m^-\boldsymbol{\Xi}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}+\mathbf{R})^{-1}\\ &= \boldsymbol{\Xi}\mathbf{P}_m^-\mathbf{H}_m^{\mathrm{T}}(\mathbf{H}\mathbf{P}_m^-\mathbf{H}_m^{\mathrm{T}}+\mathbf{R})^{-1} = \boldsymbol{\Xi}\mathbf{K}_m.\end{aligned}$$

The filter residual is $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_a = \boldsymbol{\epsilon}_m = \mathbf{y} - \mathbf{h}(\widehat{\mathbf{Q}})$, the *a posteriori* estimates are

$$\widehat{\mathbf{Q}}_m^+ = \begin{bmatrix}\mathbf{K}_m\boldsymbol{\epsilon}\\ 1\end{bmatrix} \otimes \widehat{\mathbf{Q}}^- = \boldsymbol{\Xi}\mathbf{K}_m\boldsymbol{\epsilon} + \widehat{\mathbf{Q}}^-, \text{ and}$$
$$\widehat{\mathbf{Q}}_a^+ = \widehat{\mathbf{Q}}^- + \mathbf{K}_a\boldsymbol{\epsilon} = \widehat{\mathbf{Q}}^- + \boldsymbol{\Xi}\mathbf{K}_m\boldsymbol{\epsilon} = \widehat{\mathbf{Q}}_m^+.$$

The *a posteriori* additive covariance is

$$\begin{aligned}\mathbf{P}_a^+ &= (\mathbf{I}-\mathbf{K}_a\mathbf{H}_a)\mathbf{P}_a^- = (\mathbf{I}-\boldsymbol{\Xi}\mathbf{K}_m\mathbf{H}_a)\boldsymbol{\Xi}\mathbf{P}_m^-\boldsymbol{\Xi}^{\mathrm{T}} = \boldsymbol{\Xi}(\mathbf{I}-\mathbf{K}_m\mathbf{H}_a\boldsymbol{\Xi})\mathbf{P}_m^-\boldsymbol{\Xi}^{\mathrm{T}}.\\ &= \boldsymbol{\Xi}(\mathbf{I}-\mathbf{K}_m\mathbf{H}_m)\mathbf{P}_m^-\boldsymbol{\Xi}^{\mathrm{T}} = \boldsymbol{\Xi}\mathbf{P}_m^+\boldsymbol{\Xi}^{\mathrm{T}}.\end{aligned}$$

The estimates during propagation are

$$\frac{d}{dt}\widehat{\mathbf{Q}}(t) = \frac{1}{2}\boldsymbol{\Omega}\left(\omega(t)\right)\widehat{\mathbf{Q}}(t).$$

The propagation of the covariance between measurements is given by

$$\dot{\mathbf{P}}_a(t) = \frac{1}{2}\boldsymbol{\Omega}\left(\omega(t)\right)\mathbf{P}_a(t) - \frac{1}{2}\mathbf{P}_a(t)\boldsymbol{\Omega}\left(\omega(t)\right) + \mathbf{U}_a(t); \qquad \mathbf{P}_a(t_k) = \mathbf{P}_a^+(t_k)$$
$$\dot{\mathbf{P}}_m(t) = -[\boldsymbol{\omega}(t)\times]\mathbf{P}_m(t) + \mathbf{P}_m(t)[\boldsymbol{\omega}(t)\times] + \mathbf{U}_m(t); \qquad \mathbf{P}_m(t_k) = \mathbf{P}_m^+(t_k)$$

where $\mathbf{U}_a$ and $\mathbf{U}_m$ are the spectral densities of the process noise.

It is now going to be shown that if

$$\mathbf{U}_a(t) = \boldsymbol{\Xi}\left(\widehat{\mathbf{Q}}(t)\right)\mathbf{U}_m(t)\,\boldsymbol{\Xi}^{\mathrm{T}}\left(\widehat{\mathbf{Q}}(t)\right),$$

then

$$\mathbf{P}_a(t) = \boldsymbol{\Xi}\left(\widehat{\mathbf{Q}}(t)\right)\mathbf{P}_m(t)\boldsymbol{\Xi}^{\mathrm{T}}\left(\widehat{\mathbf{Q}}(t_k)\right). \tag{3}$$

Since the equality holds at the beginning of the propagation, it is sufficient to show that both sides have the same derivative. For convenience let

$$\boldsymbol{\Xi} \triangleq \boldsymbol{\Xi}\left(\widehat{\mathbf{Q}}(t)\right), \quad [\boldsymbol{\omega}\times] \triangleq [\boldsymbol{\omega}(t)\times], \text{ and } \boldsymbol{\Omega} \triangleq \boldsymbol{\Omega}\left(\boldsymbol{\omega}(t)\right).$$

Taking the derivative of both sides of Eq. (3) yields

$$\frac{1}{2}\boldsymbol{\Omega}\,\mathbf{P}_a - \frac{1}{2}\mathbf{P}_a\boldsymbol{\Omega} + \mathbf{U}_a = \dot{\boldsymbol{\Xi}}\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}} - \boldsymbol{\Xi}[\boldsymbol{\omega}\times]\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}} + \boldsymbol{\Xi}\mathbf{P}_m[\boldsymbol{\omega}\times]\boldsymbol{\Xi}^{\mathrm{T}} + \boldsymbol{\Xi}\mathbf{U}_m\boldsymbol{\Xi}^{\mathrm{T}} + \boldsymbol{\Xi}\mathbf{P}_m\dot{\boldsymbol{\Xi}}^{\mathrm{T}} \tag{4}$$

where

$$\dot{\boldsymbol{\Xi}} = \boldsymbol{\Xi}\left(\dot{\widehat{\mathbf{Q}}}\right) = \frac{1}{2}\boldsymbol{\Xi}\left(\boldsymbol{\Omega}\,\widehat{\mathbf{Q}}\right) = \frac{1}{2}\boldsymbol{\Xi}\left(\begin{bmatrix}\widehat{\mathbf{q}}\times\boldsymbol{\omega} + \widehat{q}_4\,\boldsymbol{\omega}\\ -\boldsymbol{\omega}^{\mathrm{T}}\widehat{\mathbf{q}}\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}-\boldsymbol{\omega}^{\mathrm{T}}\widehat{\mathbf{q}}\,I_{3\times 3} + [(\widehat{\mathbf{q}}\times\boldsymbol{\omega})\times] + \widehat{q}_4[\boldsymbol{\omega}\times]\\ (\boldsymbol{\omega}\times\widehat{\mathbf{q}})^{\mathrm{T}} - \widehat{q}_4\,\boldsymbol{\omega}^{\mathrm{T}}.\end{bmatrix}$$

Utilizing the following identities

$$[(\widehat{\mathbf{q}}\times\boldsymbol{\omega})\times] = \boldsymbol{\omega}\widehat{\mathbf{q}}^{\mathrm{T}} - \widehat{\mathbf{q}}\boldsymbol{\omega}^{\mathrm{T}}, \qquad [\widehat{\mathbf{q}}\times]\cdot[\boldsymbol{\omega}\times] = \boldsymbol{\omega}\widehat{\mathbf{q}}^{\mathrm{T}} - \boldsymbol{\omega}^{\mathrm{T}}\widehat{\mathbf{q}}I_{3\times 3}$$

it follows that

$$\begin{aligned}\dot{\boldsymbol{\Xi}} &= \frac{1}{2}\begin{bmatrix}-\boldsymbol{\omega}^{\mathrm{T}}\widehat{\mathbf{q}}\,I_{3\times 3} + \boldsymbol{\omega}\widehat{\mathbf{q}}^{\mathrm{T}} - \widehat{\mathbf{q}}\boldsymbol{\omega}^{\mathrm{T}} + \widehat{q}_4[\boldsymbol{\omega}\times]\\ -\widehat{\mathbf{q}}^{\mathrm{T}}[\boldsymbol{\omega}\times] - \widehat{q}_4\,\boldsymbol{\omega}^{\mathrm{T}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\boldsymbol{\omega}^{\mathrm{T}}\widehat{\mathbf{q}}\,I_{3\times 3} + \boldsymbol{\omega}\widehat{\mathbf{q}}^{\mathrm{T}} + \widehat{q}_4[\boldsymbol{\omega}\times]\\ -\widehat{\mathbf{q}}^{\mathrm{T}}[\boldsymbol{\omega}\times]\end{bmatrix} - \frac{1}{2}\widehat{\mathbf{Q}}\,\boldsymbol{\omega}^{\mathrm{T}}\\ &= \frac{1}{2}\begin{bmatrix}-[\widehat{\mathbf{q}}\times]\cdot[\boldsymbol{\omega}\times] + \widehat{q}_4[\boldsymbol{\omega}\times]\\ -\widehat{\mathbf{q}}^{\mathrm{T}}[\boldsymbol{\omega}\times]\end{bmatrix} - \frac{1}{2}\widehat{\mathbf{Q}}\,\boldsymbol{\omega}^{\mathrm{T}} = \frac{1}{2}\boldsymbol{\Xi}[\boldsymbol{\omega}\times] - \frac{1}{2}\widehat{\mathbf{Q}}\,\boldsymbol{\omega}^{\mathrm{T}},\end{aligned} \tag{5}$$

and

$$\boldsymbol{\Omega}\,\boldsymbol{\Xi} = \begin{bmatrix}-\widehat{q}_4[\boldsymbol{\omega}\times] - [\boldsymbol{\omega}\times]\cdot[\widehat{\mathbf{q}}\times] - \boldsymbol{\omega}\widehat{\mathbf{q}}^{\mathrm{T}}\\ -\widehat{q}_4\boldsymbol{\omega}^{\mathrm{T}} - \boldsymbol{\omega}^{\mathrm{T}}[\widehat{\mathbf{q}}\times]\end{bmatrix} = -\boldsymbol{\Xi}\,[\boldsymbol{\omega}\times] - \widehat{\mathbf{Q}}\boldsymbol{\omega}^{\mathrm{T}}. \tag{6}$$

Substituting Eq. (5) in Eq. (4) yields

$$\begin{aligned}\frac{1}{2}\boldsymbol{\Omega}\,\mathbf{P}_a - \frac{1}{2}\mathbf{P}_a\boldsymbol{\Omega} &= \frac{1}{2}\boldsymbol{\Xi}[\boldsymbol{\omega}\times]\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}} - \frac{1}{2}\widehat{\mathbf{Q}}\,\boldsymbol{\omega}^{\mathrm{T}}\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}} - \boldsymbol{\Xi}[\boldsymbol{\omega}\times]\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}} + \boldsymbol{\Xi}\mathbf{P}_m[\boldsymbol{\omega}\times]\boldsymbol{\Xi}^{\mathrm{T}} - \frac{1}{2}\boldsymbol{\Xi}\mathbf{P}_m[\boldsymbol{\omega}\times]\boldsymbol{\Xi}^{\mathrm{T}} - \frac{1}{2}\boldsymbol{\Xi}\mathbf{P}_m\boldsymbol{\omega}\widehat{\mathbf{Q}}^{\mathrm{T}}\\ &= \frac{1}{2}\left\{-\boldsymbol{\Xi}[\boldsymbol{\omega}\times] - \widehat{\mathbf{Q}}\,\boldsymbol{\omega}^{\mathrm{T}}\right\}\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}} - \frac{1}{2}\boldsymbol{\Xi}\mathbf{P}_m\left\{[\boldsymbol{\omega}\times]\boldsymbol{\Xi}^{\mathrm{T}} - \boldsymbol{\omega}\widehat{\mathbf{Q}}^{\mathrm{T}}\right\}\end{aligned}$$

which is equivalent because of Eq. (6) and Eq. (3). The proof is complete.

The above arguments show that every MEKF is equivalent to an AEKF. It does not show the converse. A MEKF designed with

$$\widehat{\mathbf{Q}}_m(t_0), \qquad \mathbf{P}_m(t_0), \qquad \mathbf{R}_{m,k}, \quad \text{and } \mathbf{U}_m(t)$$

is equivalent to an AEKF with

$$\begin{aligned}&\widehat{\mathbf{Q}}_a(t_0) = \widehat{\mathbf{Q}}_m(t_0), \quad \mathbf{P}_a(t_0) = \boldsymbol{\Xi}\left(\widehat{\mathbf{Q}}_0\right)\mathbf{P}_m(t_0)\boldsymbol{\Xi}^{\mathrm{T}}\left(\widehat{\mathbf{Q}}_0\right), \quad \mathbf{R}_{a,k} = \mathbf{R}_{m,k}, \ \text{ and}\\ &\mathbf{U}_a(t) = \boldsymbol{\Xi}\left(\widehat{\mathbf{Q}}(t)\right)\mathbf{U}_m(t)\,\boldsymbol{\Xi}^{\mathrm{T}}\left(\widehat{\mathbf{Q}}(t)\right).\end{aligned}$$

Every MEKF is equivalent to an AEKF with a singular covariance matrix. This does not imply that every AEKF has a singular covariance. It is sufficient to choose a nonsingular $\mathbf{P}_a(t_0)$. It would be reasonable to ask whether the converse is true: it is possible to design an MEKF equivalent to any given (non-singular) AEKF? The answer is no.

Once more the *a posteriori* estimates are

$$\begin{aligned}\widehat{\mathbf{Q}}_m^+ &= \widehat{\mathbf{Q}}^- + \boldsymbol{\Xi}\mathbf{K}_m\boldsymbol{\epsilon}, \ \text{and}\\ \widehat{\mathbf{Q}}_a^+ &= \widehat{\mathbf{Q}}^- + \mathbf{K}_a\boldsymbol{\epsilon}.\end{aligned}$$

The filters give the same estimate if

$$\begin{aligned}&(\boldsymbol{\Xi}\mathbf{K}_m - \mathbf{K}_a)\boldsymbol{\epsilon} = 0,\\ &[\boldsymbol{\Xi}\mathbf{P}_m\mathbf{H}_m^{\mathrm{T}}(\mathbf{H}_m\mathbf{P}_m\mathbf{H}_m^{\mathrm{T}} + \mathbf{R})^{-1} - \mathbf{P}_a\mathbf{H}_a^{\mathrm{T}}(\mathbf{H}_a\mathbf{P}_a\mathbf{H}_a^{\mathrm{T}} + \mathbf{R})^{-1}]\boldsymbol{\epsilon} = 0, \ \text{and}\\ &[\boldsymbol{\Xi}\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}}\mathbf{H}_a^{\mathrm{T}}(\mathbf{H}_a\boldsymbol{\Xi}\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}}\mathbf{H}_a^{\mathrm{T}} + \mathbf{R})^{-1} - \mathbf{P}_a\mathbf{H}_a^{\mathrm{T}}(\mathbf{H}_a\mathbf{P}_a\mathbf{H}_a^{\mathrm{T}} + \mathbf{R})^{-1}]\boldsymbol{\epsilon} = 0.\end{aligned}$$

Unless the residuals have an unusual structure, it is impossible that every realization belongs to the null space of the same matrix, therefore the term in brackets must be zero. If $\mathrm{rank}(\mathbf{P}_a) = 4$

$$\boldsymbol{\Xi}\mathbf{P}_m\boldsymbol{\Xi}^{\mathrm{T}} \neq \mathbf{P}_a \quad \forall\mathbf{P}_m.$$

The following equation should be considered

$$\mathbf{\Xi P}_m\mathbf{\Xi}^{\mathrm{T}}\mathbf{H}_a^{\mathrm{T}}(\mathbf{H}_a\mathbf{\Xi P}_m\mathbf{\Xi}^{\mathrm{T}}\mathbf{H}_a^{\mathrm{T}}+\mathbf{R})^{-1}=\mathbf{P}_a\mathbf{H}_a^{\mathrm{T}}(\mathbf{H}_a\mathbf{P}_a\mathbf{H}_a^{\mathrm{T}}+\mathbf{R})^{-1}. \tag{7}$$

Solving Eq. (7) is not always possible. Assume, for example, that the initial orientation is along the reference frame and the initial covariance is

$$\mathbf{P}_a=\kappa\mathbf{I}_{4\times 4};\qquad \mathbf{\Xi}=\begin{bmatrix}\mathbf{I}_{3\times 3}\\ \mathbf{O}_{1\times 3}\end{bmatrix},$$

where κ is a given positive number. Then Eq. (7) becomes

$$\begin{bmatrix}\mathbf{P}_m & \mathbf{O}_{3\times 1}\\ \mathbf{O}_{1\times 3} & 0\end{bmatrix}\mathbf{H}_a^{\mathrm{T}}\left(\mathbf{H}_a\begin{bmatrix}\mathbf{P}_m & \mathbf{O}_{3\times 1}\\ \mathbf{O}_{1\times 3} & 0\end{bmatrix}\mathbf{H}_a^{\mathrm{T}}+\mathbf{R}\right)^{-1}=\kappa\mathbf{H}_a^{\mathrm{T}}(\kappa\mathbf{H}_a\mathbf{H}_a^{\mathrm{T}}+\mathbf{R})^{-1}.$$

It can be seen that no $\mathbf{P}_m$ will work because the fourth row of the left side of the equation is always going to be zero.

In summary, every design of a MEKF corresponds to a singular AEKF, while not every non-singular AEKF corresponds to a MEKF. Researchers agree that the covariance of the AEKF does not need to be strictly singular[4]. Therefore, there are many possible AEKFs that are not equivalent to a corresponding MEKF. Both approaches should be explored in searching for the best estimator for an attitude estimation problem.

B. Normalization

Both multiplicative and additive approaches provide estimates with unit norm to first order[16] with respect to the estimation error. It will be proven that in the multiplicative approach the *a posteri* estimate norm is always greater than the *a priori* norm, and that the estimate norm is unchanged through propagation. As a consequence, brute force normalization is essential in the multiplicative update to avoid the norm of the estimate becoming arbitrarily large. Of course this can be avoided by using the full nonlinear transformation between the three-dimensional representation of the attitude error and the quaternion. The downside would be that the estimate will depend on the parametrization (rotation vector, Gibbs vector, modified Rodrigues parameters) which is counterintuitive.

The *a posteriori* estimate is given by

$$\widehat{\mathbf{Q}}^{+}=\delta\widehat{\mathbf{Q}}\otimes\widehat{\mathbf{Q}}^{-}=\begin{bmatrix}\delta\widehat{q}_4\widehat{\mathbf{q}}^{-}-[\delta\widehat{\mathbf{q}}\times]\widehat{\mathbf{q}}^{-}+\widehat{q_4^{-}}\,\delta\widehat{\mathbf{q}}\\ -\delta\widehat{\mathbf{q}}^{\mathrm{T}}\widehat{\mathbf{q}}^{-}+\delta\widehat{q}_4\widehat{q_4^{-}}\end{bmatrix}.$$

The norm can be computed to be

$$\|\widehat{\mathbf{Q}}^{+}\|^2=\|\widehat{\mathbf{Q}}^{-}\|^2\cdot\|\delta\widehat{\mathbf{Q}}\|^2,\ \text{and}$$

$$\|\delta\widehat{\mathbf{Q}}\|^2=1+\|\delta\widehat{\mathbf{q}}\|^2>1.$$

Therefore, the *a posteriori* estimate norm is always greater than the *a priori* estimate norm. During propagation, the norm remains unchanged since the quadratic form of a skew-symmetric matrix is always zero,

$$\frac{d}{dt}\|\mathbf{Q}\|^2=\frac{d}{dt}\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right)=2\mathbf{Q}^{\mathrm{T}}\dot{\mathbf{Q}}=2\mathbf{Q}^{\mathrm{T}}\mathbf{\Omega}(\omega)\mathbf{Q}=0.$$

The square of the norm remaining constant implies that the norm remains constant since the norm is always positive.

In summary, it was shown that at every update the norm of the estimate increases, while during propagation the estimate norm remains the same. Hence the norm of the estimate of a MEKF will constantly increase. To avoid this situation, it is necessary to normalize the estimate.

The rate at which the estimate norm increases can be reduced by using the second order MEKF, in which

$$\delta\widehat{\mathbf{Q}}=\begin{bmatrix}\delta\widehat{\mathbf{q}}\\ 1-\|\delta\widehat{\mathbf{q}}\|^2/2\end{bmatrix}.$$

The norm is still greater than one,

$$\|\delta\widehat{\mathbf{Q}}\|^2 = \|\delta\widehat{\mathbf{q}}\|^2 + 1 + \|\delta\widehat{\mathbf{q}}\|^4/4 - \|\delta\widehat{\mathbf{q}}\|^2 = 1 + \|\delta\widehat{\mathbf{q}}\|^4/4 > 1.$$

Therefore normalization is still necessary. In the MEKF, the error on the norm is of first order (or second), but always positive, and the cumulative effect after many updates could result in large deviations from unitary norm. In the AEKF, the error in the norm could be either positive or negative, making the normalization necessary but less crucial after some time. Figure 2 shows the evolution of the norm in the additive and multiplicative case if the normalization was not enforced after each update. Estimating the quaternion without normalization is not recommended.[15] However, from Fig. 2 it should be clear that normalization is an essential part of the MEKF as it is for the AEKF.

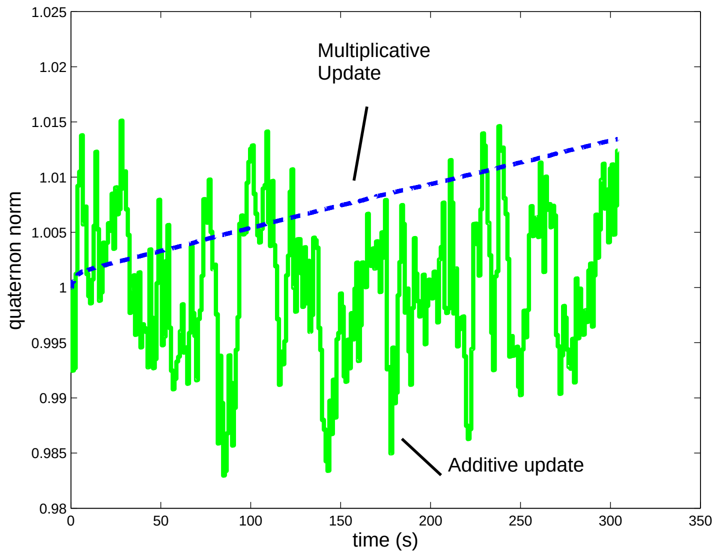

Figure 2. Norm evolution without brute force normalization. Dashed line is the multiplicative update, continuous line is the additive update.

Theoretically, the norm constraint could be enforced with a perfect measurement of the state norm. A perfect measurement would produce a singular covariance matrix, where the singularity is a byproduct of the linearization process[4] and does not signify that the covariance in the AEKF is singular.

The Mean Square Error is the standard way of evaluating estimators. Having a small mean square error is desirable, independent of the possible physical interpretation. It was also shown how a more physically meaningful definition of the attitude estimation error leads to a Kalman filter formulation equivalent to the one defining the error in standard way. Arguments were presented to provide an explanation of the fact that in practical applications, the covariance of the AEKF does not become ill-conditioned when process noise is present. Finally, it was shown that brute force normalization is fundamental in both schemes. A justification of brute force normalization will be presented next.

V. Norm Constrained Kalman Filtering

In this section, it will be shown that brute force normalization is *optimal* in a MSE sense, not only in a geometrical sense as previously shown.[6] Normalization is a nonlinear transformation, therefore similar approximations to those associated with the extended Kalman filter will be made. Optimality does not hold strictly, but conditionally on the above approximations.

Define the *a priori* state estimate $\widehat{\mathbf{x}}_k^-$ to be the state estimate at time t_k just prior to employing the measurement $\mathbf{y}_k$ in the state estimate update algorithm. Similarly, define the *a posteriori* state estimate $\hat{\mathbf{x}}_k^+$ to be the state estimate at time t_k just after the state estimate update. The performance index is defined as

$$\mathcal{J}_k = E\left[\left(\mathbf{e}_k^+\right)^{\mathrm{T}}\mathbf{e}_k^+\right] \tag{8}$$

where the *a priori* and *a posteriori* estimation errors are given by

$$\mathbf{e}_k^- = \mathbf{x}_k - \widehat{\mathbf{x}}_k^-, \text{ and}$$
$$\mathbf{e}_k^+ = \mathbf{x}_k - \widehat{\mathbf{x}}_k^+ \ ,$$

respectively. Associated with the estimation errors, we can define the matrices

$$\mathbf{P}_k^- = E\left[\mathbf{e}_k^- \left(\mathbf{e}_k^-\right)^{\mathrm{T}}\right], \text{ and}$$
$$\mathbf{P}_k^+ = E\left[\mathbf{e}_k^+ \left(\mathbf{e}_k^+\right)^{\mathrm{T}}\right] \ , \tag{9}$$

before and after the measurement update, respectively. Note that

$$\mathcal{J}_k = \operatorname{trace} \mathbf{P}_k^+ \ .$$

The norm of the state vector is desired to have a predefined value

$$\|\widehat{\mathbf{x}}_k^+\| = \sqrt{l}.$$

This constraint is equivalent to the following scalar quadratic representation

$$(\widehat{\mathbf{x}}_k^+)^{\mathrm{T}}\widehat{\mathbf{x}}_k^+ = l. \tag{10}$$

The update is

$$\widehat{\mathbf{x}}_k^+ = \widehat{\mathbf{x}}_k^- + \mathbf{K}_k\boldsymbol{\epsilon}_k,$$

where $\boldsymbol{\epsilon}_k = \mathbf{y}_k - \widehat{\mathbf{y}}_k$ is the residual. Substituting the residual into Eq. (10), the state constraint can be expressed more conveniently as a control constraint:

$$\boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}}\mathbf{K}_k\boldsymbol{\epsilon}_k + 2\widehat{\mathbf{x}}_k^{\mathrm{T}}K_k\boldsymbol{\epsilon}_k + \widehat{\mathbf{x}}_k^{\mathrm{T}}\widehat{\mathbf{x}}_k - l = 0. \tag{11}$$

The goal is to find the gain $\mathbf{K}_k$ such that Eq. (8) is minimized and the constraint given by Eq. (11) is satisfied.

A. First Order Condition

The *a posteriori* error mean square is given by [†]

$$\mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^{\mathrm{T}}$$

where $\mathbf{P}_k^-$ is the *a priori* state error mean square. Define

$$\mathbf{W}_k \triangleq \mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k.$$

Therefore the Joseph formula can be rewritten as

$$\mathbf{P}_k^+ = \mathbf{P}_k^- - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}} + \mathbf{K}_k\mathbf{W}_k\mathbf{K}_k^{\mathrm{T}}.$$

The performance index to be minimized is then given by

$$\mathcal{J}_k = \operatorname{trace}\left[\mathbf{P}_k^- - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}} + \mathbf{K}_k\mathbf{W}_k\mathbf{K}_k^{\mathrm{T}}\right].$$

The Kalman gain should be computed to satisfy the constraint in Eq. (11). Matrix $\mathbf{P}_k^-$ is $n \times n$, $\mathbf{K}_k$ is $m \times n$, l is a scalar and the remaining are of appropriate dimensions. The augmented performance index is

$$\mathcal{J}_k = \operatorname{trace}\left[\mathbf{P}_k^- - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}} + \mathbf{K}_k\mathbf{W}_k\mathbf{K}_k^{\mathrm{T}}\right] + \lambda_k\left[\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}}\widehat{\mathbf{x}}_k^- + 2\boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}}\widehat{\mathbf{x}}_k^- + \boldsymbol{\epsilon}_k^{\mathrm{T}}\mathbf{K}_k^{\mathrm{T}}\mathbf{K}_k\boldsymbol{\epsilon}_k - l\right].$$

The $n \times m + 1$ optimal values of λ_k and $\mathbf{K}_k$ are obtained solving the $n \times m$ equations resulting from taking the derivative of $\mathcal{J}_k$ with respect to $\mathbf{K}_k$ and setting it to zero

$$-2\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + 2\mathbf{K}_k\mathbf{W}_k + 2\lambda_k(\widehat{\mathbf{x}}_k^-\boldsymbol{\epsilon}_k^{\mathrm{T}} + \mathbf{K}_k\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_k^{\mathrm{T}}) = 0, \tag{12}$$

[†] see page 13.

and the scalar constraint (11). The vector and matrix derivatives used to obtain Eq. (12) are listed in the appendix.

Equation (12) can be rewritten to obtain the following first order conditions

$$\mathbf{K}_k = (\mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} - \lambda_k \widehat{\mathbf{x}}_k^- \boldsymbol{\epsilon}_k^{\mathrm{T}})(\mathbf{W}_k + \lambda_k \boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^{\mathrm{T}})^{-1}, \text{ and}$$
$$\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{K}_k^{\mathrm{T}} \mathbf{K}_k \boldsymbol{\epsilon}_k + 2\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \mathbf{K}_k \boldsymbol{\epsilon}_k + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \left(\widehat{\mathbf{x}}_k^-\right) - l = 0.$$

Using the matrix inversion lemma (see Appendix), it follows that

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} - \lambda_k \widehat{\mathbf{x}}_k^- \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} - \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \frac{\lambda_k \boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1}}{1 + \lambda_k \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k} + \lambda_k \widehat{\mathbf{x}}_k^- \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \frac{\lambda_k \boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1}}{1 + \lambda_k \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k}.$$

Substituting into Eq. (11), after some manipulations, the following scalar equation with the scalar unknown λ_k is obtained

$$\lambda_k^2 \widetilde{\epsilon}_k^2 \left(-\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \widehat{\mathbf{x}}_k^- + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \left(\widehat{\mathbf{x}}_k^-\right) - l\right) + \lambda_k \widetilde{\epsilon}_k \left(-2\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \widehat{\mathbf{x}}_k^- + 2\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \left(\widehat{\mathbf{x}}_k^-\right) - 2l\right) +$$
$$+ \left(\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \mathbf{H}_k \mathbf{P}_k^- \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k + 2\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \left(\widehat{\mathbf{x}}_k^-\right) - l\right) = 0,$$

where

$$\widetilde{\epsilon}_k = \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k.$$

Therefore, the optimal lagrange multiplier is

$$\lambda_k = \frac{-b/2 \pm \sqrt{b^2/4 - ac}}{a},$$

where

$$a = -l\widetilde{\epsilon}_k^2, \qquad b = -2\widetilde{\epsilon}_k l, \text{ and}$$
$$c = \boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \mathbf{H}_k \mathbf{P}_k^- \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k + 2\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \left(\widehat{\mathbf{x}}_k^-\right) - l.$$

Finally, it follows that

$$\lambda_k = \frac{\widetilde{\epsilon}_k l \pm \sqrt{\widetilde{\epsilon}_k^2 l^2 + l\widetilde{\epsilon}_k^2 c}}{-l\widetilde{\epsilon}_k^2} = \frac{1 \pm \sqrt{1 + c/l}}{-\widetilde{\epsilon}_k}. \tag{13}$$

Notice that

$$1 + c/l = (\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \mathbf{H}_k \mathbf{P}_k^- \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k + 2\left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}} \left(\widehat{\mathbf{x}}_k^-\right))/l$$
$$= (\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \mathbf{H}_k \mathbf{P}_k^- + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}})^{\mathrm{T}} (\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \mathbf{H}_k \mathbf{P}_k^- + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}})/l \geq 0$$

Therefore, λ_k is always a real number and can be rewritten as

$$\lambda_k = \frac{-1}{\widetilde{\epsilon}_k} \pm \frac{\|\widehat{\mathbf{x}}_k^- + \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \boldsymbol{\epsilon}_k\|}{\widetilde{\epsilon}_k \sqrt{l}}.$$

B. Second Order Condition

Taking the second derivative of the performance index presents some representation issues. Each of the entrees of the first derivative could be differentiated again, but this approach will result in $m \times n$ matrix equations. Another approach would be to perturb the gain and show that the perturbation results in an increment of the performance index.

In the case of scalar measurement, the gain $\mathbf{K}_k$ can be partitioned as

$$\mathbf{K}_k = \begin{bmatrix} \mathbf{k}_k \\ k_k \end{bmatrix},$$

where k_k is a scalar. The constraint becomes

$$\epsilon_k^2 \mathbf{k}_k^{\mathrm{T}} \mathbf{k}_k + \epsilon_k^2 k_k^2 + 2\epsilon_k \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} \mathbf{k}_k + 2\epsilon_k x_k^- k_k + \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} \widehat{\boldsymbol{\chi}}_k^- - l = 0,$$

where

$$\widehat{\mathbf{x}}_k^- = \begin{bmatrix} \widehat{\boldsymbol{\chi}}_k^- \\ \widehat{x}_k^- \end{bmatrix}.$$

Differentiating the constraint, yields

$$2(\epsilon_k^2 \mathbf{k}_k^{\mathrm{T}} + \epsilon_k \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}}) d\mathbf{k}_k + 2(\epsilon_k^2 k_k + \epsilon_k \widehat{x}_k^-) dk_k = 0.$$

Assuming the residual is not zero (if the residual is zero the *a posteriori* estimate is always equal to the *a priori* estimate), it follows that

$$dk_k = -\frac{\epsilon_k \mathbf{k}_k^{\mathrm{T}} + \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}}}{\epsilon_k k_k + \widehat{x}_k^-} d\mathbf{k}_k.$$

The second-order differential is

$$\begin{aligned} d\mathcal{J}_k^2 &= d\mathbf{K}_k^{\mathrm{T}} G_{KK} d\mathbf{K}_k, \\ G_{KK} &= 2W_k + 2\lambda_k \epsilon_k^2, \text{ and} \\ d\mathcal{J}_k^2 &= (2W_k + 2\lambda_k \epsilon_k^2)(d\mathbf{K}_k^T d\mathbf{K}_k) = (2W_k + 2\lambda_k \epsilon_k^2)(d\mathbf{k}_k^T d\mathbf{k}_k + dk_n^2) \\ &= 2(W_k + \lambda_k \epsilon_k^2) d\mathbf{k}_k^{\mathrm{T}} \left(\mathbf{I} + \frac{\epsilon_k \mathbf{k}_k + \widehat{\boldsymbol{\chi}}_k^-}{\epsilon_k k_k + \widehat{x}_k^-} \frac{\epsilon_k \mathbf{k}_k^{\mathrm{T}} + \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}}}{\epsilon k_k + \widehat{x}_k^-} \right) d\mathbf{k}_k. \end{aligned}$$

The sufficient condition for a minimum is

$$\frac{2(W_k + \lambda_k \epsilon_k^2)}{(\epsilon_k k_k + \widehat{x}_k^-)^2} \left[(\epsilon_k k_k + \widehat{x}_k^-)^2 \mathbf{I} + \epsilon_k^2 \mathbf{k}_k \mathbf{k}_k^{\mathrm{T}} + \epsilon_k \widehat{\boldsymbol{\chi}}_k^- \mathbf{K}_k^{\mathrm{T}} + \epsilon_k \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} + \epsilon_k \mathbf{k}_k \widehat{\boldsymbol{\chi}}_k^- \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} \right] > 0$$

$$\mathbf{k}_k = \frac{\left(\widetilde{\mathbf{P}}_k^-\right)^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} - \lambda_k \epsilon_k \widehat{\boldsymbol{\chi}}_k^-}{W_k + \lambda_k \epsilon_k^2}, \qquad k_k = \frac{\mathbf{p}^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} - \lambda_k \epsilon_k \widehat{x}_k^-}{W_k + \lambda_k \epsilon_k^2}, \qquad \mathbf{P}_k^- = \begin{bmatrix} \widetilde{\mathbf{P}}_k^- & \mathbf{p}_k \end{bmatrix}.$$

Substituting in the gain and eliminating positive scalars, yields

$$\begin{aligned} &(W_k + \lambda_k \epsilon_k^2) \Bigg\{ (\epsilon_k k_k + \widehat{x}_k^-)^2 \mathbf{I} + \frac{\epsilon_k^2}{(W_k + \lambda_k \epsilon_k^2)^2} \left[\left(\widetilde{\mathbf{P}}_k^-\right)^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} \mathbf{H}_k \widetilde{\mathbf{P}}_k^- - \lambda_k \epsilon_k \widehat{\boldsymbol{\chi}}_k^- \mathbf{H}_k \widetilde{\mathbf{P}}_k^- - \lambda_k \epsilon_k \left(\widetilde{\mathbf{P}}_k^-\right)^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} \right. \\ &\left. + \lambda_k^2 \epsilon_k^2 \widehat{\boldsymbol{\chi}}_k^- \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} \right] + \frac{r}{W_k + \lambda_k \epsilon_k^2} \left[\left(\widetilde{\mathbf{P}}_k^-\right)^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} + \widehat{\boldsymbol{\chi}}_k^- \mathbf{H}_k \widetilde{\mathbf{P}}_k^- - 2\lambda_k \epsilon \widehat{\boldsymbol{\chi}}_k^- \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} \right] + \widehat{\boldsymbol{\chi}}_k^- \left(\widehat{\boldsymbol{\chi}}_k^-\right)^{\mathrm{T}} \Bigg\} > 0. \end{aligned}$$

An equivalent condition is

$$(W_k + \lambda_k \epsilon_k^2) \left\{ (\epsilon_k k_k + \widehat{x}_k^-)^2 (W_k + \lambda_k \epsilon_k^2)^2 \mathbf{I} + \left[\epsilon_k \left(\widetilde{\mathbf{P}}_k^-\right)^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} + W_k \widehat{\boldsymbol{\chi}}_k^- \right] \left[\epsilon_k \left(\widetilde{\mathbf{P}}_k^-\right)^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} + W_k \widehat{\boldsymbol{\chi}}_k^- \right]^{\mathrm{T}} \right\} > 0. \tag{14}$$

The matrix in brackets is of the form

$$\mu^2 \mathbf{I} + \mathbf{v}\mathbf{v}^{\mathrm{T}},$$

which is positive definite when $\mu \neq 0$. As a consequence, the optimal gain produces a minimum performance index when the scalar $W_k + \lambda_k \epsilon_k^2$ is positive. Since

$$W_k + \lambda_k \epsilon_k^2 = \pm \sqrt{W_k^2 + c/l},$$

the minimum occurs when the plus sign is chosen for the lagrange multiplier. Also, if the minus sign is chosen, the performance index will be maximized. The same arguments hold true when the measurement is a vector.

C. Constrained Minimum Solution

The performance index is minimized and the constraint is satisfied when the optimal gain is chosen as

$$\mathbf{K}_k^* = (\mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} - \lambda_k \widehat{\mathbf{x}}_k^- \boldsymbol{\epsilon}_k^{\mathrm{T}})(\mathbf{W}_k + \lambda_k \boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^{\mathrm{T}})^{-1}, \text{ where}$$
$$\lambda_k = \frac{-1}{\widetilde{\epsilon}_k} + \frac{\|\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1} \mathbf{H}_k \mathbf{P}_k^- + \left(\widehat{\mathbf{x}}_k^-\right)^{\mathrm{T}}\|}{\widetilde{\epsilon}_k \sqrt{l}}.$$

The asterisk was added to distinguish from the unconstrained Kalman gain $\mathbf{K}_k$

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k \mathbf{W}_k^{-1},$$

the unconstrained *a posteriori* estimate is $\widehat{\mathbf{x}}_k^+$

$$\widehat{\mathbf{x}}_k^+ = \widehat{\mathbf{x}}_k^- + \mathbf{K}_k \boldsymbol{\epsilon}_k.$$

The minimizing constrained gain can be rewritten as

$$\mathbf{K}_k^* = \mathbf{K}_k + \left(\frac{\sqrt{l}}{\|\widehat{\mathbf{x}}_k^+\|} - 1\right) \widehat{\mathbf{x}}_k^+ \frac{\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1}}{\widetilde{\epsilon}_k}.$$

Property 1. *The optimal constrained solution shares the same direction as the optimal unconstrained solution.*

Proof. Let $\widehat{\mathbf{x}}_k^*$ be the optimal constrained estimate. Then it follows that

$$\widehat{\mathbf{x}}_k^* = \widehat{\mathbf{x}}_k^- + \mathbf{K}_k^* \boldsymbol{\epsilon}_k = \widehat{\mathbf{x}}_k^- + \mathbf{K}_k \boldsymbol{\epsilon}_k + \left(\frac{\sqrt{l}}{\|\widehat{\mathbf{x}}_k^+\|} - 1\right) \widehat{\mathbf{x}}_k^+ \frac{\boldsymbol{\epsilon}_k^{\mathrm{T}} \mathbf{W}_k^{-1}}{\widetilde{\epsilon}_k} \boldsymbol{\epsilon}_k = \frac{\sqrt{l}}{\|\widehat{\mathbf{x}}_k^+\|} \widehat{\mathbf{x}}_k^+.$$

□

So $\widehat{\mathbf{x}}_k^*$ and $\widehat{\mathbf{x}}_k^+$ have the same direction, but different magnitude. Property 1 states that brute force normalization is optimal not only in a geometrical sense, but also in a Mean Square Error sense.

The *a posteriori* estimation error is

$$\mathbf{e}^* = (I - \mathbf{K}_k^* \mathbf{H})\mathbf{e}^- + \mathbf{K}_k^* \mathbf{v}_k.$$

Under the assumption that measurement noise is independent of process noise and initial estimation error, we have

$$\mathbf{P}_k^* = E[(\mathbf{I} - \mathbf{K}_k^* \mathbf{H}_k)\mathbf{e}^- (\mathbf{e}^-)^{\mathrm{T}} (\mathbf{I} - \mathbf{K}_k^* \mathbf{H}_k)^{\mathrm{T}}] + E[\mathbf{K}_k^* \mathbf{v}_k \mathbf{v}_k^{\mathrm{T}} (K^*)^{\mathrm{T}}]. \quad (15)$$

The optimal gain is a function of the *a priori* state and the residual, therefore it is a random variable and it should not be taken out the expectation operator. A similar situation happens in nonlinear Kalman filtering. In the extended Kalman filter, for example, the measurement mapping matrix is a function of the *a priori* state, thus making the gain a function of the *a priori* state as well. In this work, the Kalman gain is taken out of the expectation sign, following the EKF solution

$$\mathbf{P}_k^* = (\mathbf{I} - \mathbf{K}_k^* \mathbf{H}_k)\mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k^* \mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k^* \mathbf{R}_k \left(\mathbf{K}_k^*\right)^{\mathrm{T}}.$$

Substituting for $\mathbf{K}_k^*$ yields

$$\mathbf{P}_k^* = \mathbf{P}_k^+ + \frac{1}{\widetilde{\epsilon}_k} \left(1 - \frac{\sqrt{l}}{\|\widehat{\mathbf{x}}_k^+\|}\right)^2 \widehat{\mathbf{x}}_k^+ \left(\widehat{\mathbf{x}}_k^+\right)^{\mathrm{T}}, \quad (16)$$

which is very similar to the correction given by Choukroun *et al.*[18]

When two random variables are related through a nonlinear transformation, it is generally impossible to relate exclusively their second moments but all the moments of the original variable will contribute in the second moment of the transformed variable. Therefore, the correction on the covariance can be accurate or not depending on the distribution. Both the AEKF and MEKF provide estimates with unit norm to first order,[16] therefore the regular covariance $\mathbf{P}_k^+$ is an approximation accurate to first order. If the AEKF necessitates a covariance adjustment, so does the first order MEKF since brute force normalization affects the multiplicative error as well.

Matrix $\mathbf{P}_k^*$ is an approximation, and like any approximation might not be satisfactory under certain circumstances. From Eq. (16) it can be seen that P_k^* can be unsatisfactory for small $\widetilde{\epsilon}_k$ and large norm errors of the unconstrained estimate. This situation could arise, for example, in the presence of scalar measurement when the estimation error is large.

VI. Simulation Results

In this section the norm constrained estimation algorithm is tested in a spacecraft attitude simulation. The *true* quaternion evolution is obtain from a high-fidelity simulation, together with the angular velocity. The angular velocity is corrupted with scale and misalignment factors together with white noise and a bias to simulate a gyro measurement. The gyro measurement is available at 10Hz and is used to propagate the state. Two vector measurements are assumed to be available, each one is modelled as[19]

$$\mathbf{y}_i = T(\mathbf{Q})\mathbf{r}_i + \mathbf{u}_i$$

where $\mathbf{u}_i$ is the sensor noise that satisfies

$$E\left\{\mathbf{u}_i\right\} = \mathbf{0}, \qquad E\left\{\mathbf{u}_j\mathbf{u}_i^{\mathrm{T}}\right\} = \sigma_i^2\mathbf{I}_{3\times 3}\,\delta_{ij}.$$

Simulation results show that the filter *covariance* is not singular and that matches well the estimation error. Also the estimate has the desired norm at all times. Figure 3 shows the simulation results.

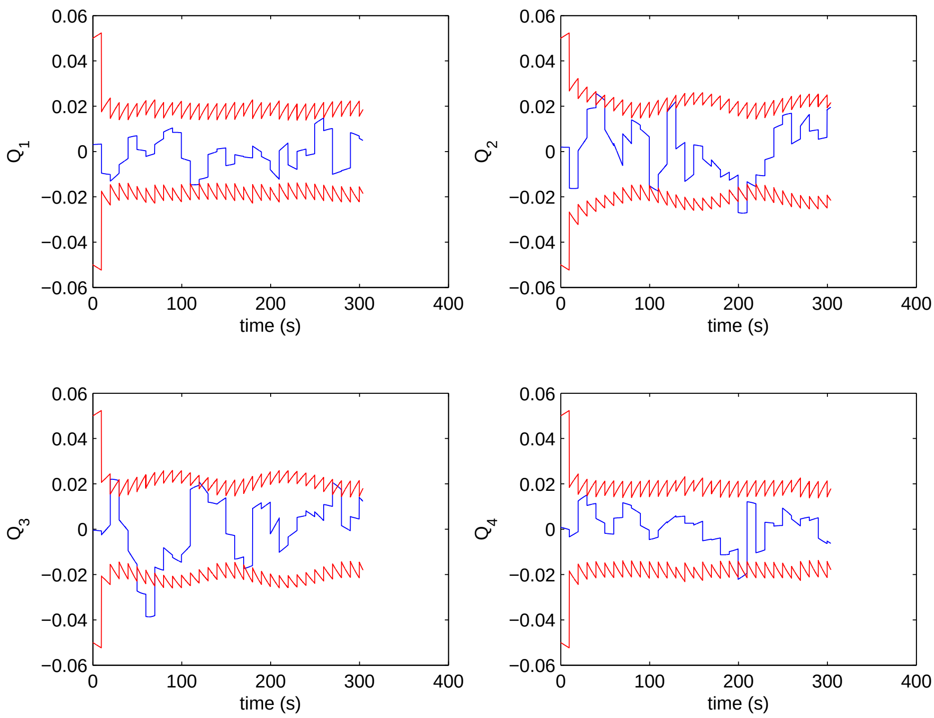

Figure 3. Error evolution of the norm constrained quaternion filter.

VII. Conclusion

It was shown that the AEKF does not need to possess a singular covariance and that brute force normalization is optimal under standard nonlinear filtering assumptions. The multiplicative approach has the advantages of a covariance with smaller dimension and easy physical interpretation of the error, but is otherwise not superior to the additive approach.

Appendix

The following calculus identities were used in the derivations:

$$
\begin{aligned}
&d/d\mathbf{X}\ (\mathbf{a}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{b}) = \mathbf{b}\mathbf{a}^{\mathrm{T}} \\
&d/d\mathbf{X}\ (\mathbf{a}^{\mathrm{T}}\mathbf{X}\mathbf{b}) = \mathbf{a}\mathbf{b}^{\mathrm{T}} \\
&d/d\mathbf{X}\ (\mathbf{a}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{X}\mathbf{b}) = \mathbf{X}(\mathbf{a}^{\mathrm{T}}\mathbf{b} + \mathbf{b}\mathbf{a}^{\mathrm{T}}) \\
&d/d\mathbf{X}\ (\mathrm{trace}(\mathbf{A}^{\mathrm{T}}\mathbf{X}\mathbf{B}^{\mathrm{T}})) = d/d\mathbf{X}(\mathrm{trace}(\mathbf{B}\mathbf{X}^{\mathrm{T}}\mathbf{A})) = \mathbf{A}\mathbf{B} \\
&d/d\mathbf{X}\ (\mathrm{trace}(\mathbf{X}\mathbf{A}\mathbf{X}^{\mathrm{T}})) = \mathbf{X}(\mathbf{A} + \mathbf{A}^{\mathrm{T}}).
\end{aligned}
$$

Capital bold letters indicate matrices, lowercase bold indicates column vector. The matrix inversion lemma was also used.

$$
if\ \ det(\mathbf{C} + \mathbf{VAU}) \neq 0 \Rightarrow (\mathbf{A}^{-1} + \mathbf{U}\mathbf{C}^{-1}\mathbf{V})^{-1} = \mathbf{A} - \mathbf{AU}(\mathbf{C} + \mathbf{VAU})^{-1}\mathbf{VA}
$$

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