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Precision Entry Navigation Dead-Reckoning Error Analysis: Theoretical Foundations of the Discrete-Time Case

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A linear covariance analysis strategy is developed for application to hypersonic atmospheric planetary entry where the only available navigation data is provided by strapdown inertial measurement units. The navigation scenario considered encompasses the so-called dead-reckoning navigation, wherein the inertial measurement unit provides measures of the change in velocity and the change in attitude at discrete times. These measurements are used to propagate an initial state estimate forward in time. The question that is addressed is quantifying the accuracy of the state estimate using dead-reckoning during a typical Mars atmospheric entry. The inertial measurement data is assumed to be corrupted with random noise, random constant biases, misalignment errors, and scale factor errors. The location of the inertial measurement unit with respect to the spacecraft's center of mass is also considered to contain uncertainty.

INTRODUCTION

The work presented here is a significant expansion of the work presented by Crain and Bishop [1]. In [1], the authors considered continuous-time inertial measurement unit (IMU) measurements and modeled the attitude errors using additive quaternion errors. In this paper, the more realistic situation of discrete-time IMU measurements are considered and the attitude errors are modeled using multiplicative quaternion error methods.

During precision entry, descent, and landing (EDL) at Mars, various classes of measurements become available. Strapdown IMU data (measurements of non-gravitational acceleration and angular velocity) are always available during EDL. Other measurements typically become available after heat shield jettison. External measurements include radar altimetry, LIDAR, and optical and radiometric observations. The lack of external measurements in the upper atmosphere is the main difficulty in achieving precision landing on Mars. In the upper atmosphere, the guidance depends on IMU dead-reckoning navigation. Once the heat shield is jettisoned and external measurements become available, the navigation uncertainty is greatly reduced, but the the maneuverability of the vehicle is compromised by the hypersonic parachute. After the parachutes are jettisoned, the altimeter and LIDAR become available, and guidance can actively maneuver the vehicle. However, there may not be enough time to make large corrections and achieve pinpoint landing.

The goal of this investigation is to provide the navigated state and an accurate representation of its uncertainty during the entry phase in which IMU measurements are the only data available. The navigation architecture uses the conventional approach of propagating the estimated state using the IMU data, known as dead-reckoning. This work addresses the problems of correctly incorporating various sources of IMU errors into the estimated error covariance. Correct knowledge of estimated state uncertainty is important to GNC because active guidance acts on the navigated state.

The IMU unit contains both a gyro and an accelerometer package. The gyros provide measurements of the integral of the relative angular velocity of the IMU case reference frame relative to the

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inertial reference frame, denoted by $\Delta\boldsymbol{\theta}^c$. The IMU is installed at a location offset from the center of mass. This relative position is not known perfectly and is subject to change as the spacecraft's center of mass changes due to fuel consumption, heat shield jettison, and parachute deployment and jettison. The offset of the IMU from the center of mass is accounted for in this investigation, and its uncertainty is included in the estimation error covariance. The navigation state is comprised of position and velocity of the IMU. This approach gives better performance than estimating position and velocity of the vehicle center of mass since the gravitational acceleration is less sensitive to small errors in position than is the non-gravitational acceleration.

The effects of IMU biases are included in the covariance analysis. The most straightforward technique to include biases in the Kalman filter is to augment the state vector and estimate the biases. In an attempt to decouple the bias estimation from the state estimation, Friedland estimated the state as though the bias was not present, and then added the contribution of the bias. Friedland showed [2] that this approach is equivalent to augmenting the state vector. This technique, known as two-stage Kalman filter or separate-bias Kalman estimation, was then extended to incorporate a walk in the bias forced by white noise [3]. To account for the bias walk, the process noise covariance was increased heuristically, and optimality conditions were derived [4, 5]. During the Mars entry phase, the lack of external measurements prevents the use of these techniques.

In this work, a completely different approach is taken. The effects of the constant random bias in the Kalman filter are considered as an error and not as a state. This is important, for example, when the bias is not observable, or when there is not enough information to discern the bias from the measurements. When this situation arises, the classical approach is to tune the filter such that the sample covariance obtained through Monte Carlo analysis matches the predicted covariance. The technique presented here is useful in quantifying the uncertainty due to a random bias in a single run, which would aid in tuning the filter.

The approach taken is different from that of the consider filter [6, 7]. The consider filter can be designed to solve the same problem, and the two algorithms, although different, are equivalent. The next section will introduce the general algorithm to be applied to Mars EDL navigation.

DISCRETE KALMAN FILTER WITH UNCOMPENSATED BIAS

Consider the stochastic system of difference equations

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{\Upsilon}_k \mathbf{b}_\nu + \boldsymbol{\nu}_k,$$

where $\boldsymbol{\nu}_k$ is process noise assumed to be a zero-mean, white noise sequence with

$$\mathrm{E}\left\{\boldsymbol{\nu}_k\right\} = \mathbf{0} \;\; \forall\, k, \qquad \mathrm{E}\left\{\boldsymbol{\nu}_j\, \boldsymbol{\nu}_k^{\mathrm{T}}\right\} = \mathbf{Q}_k\, \delta_{jk},$$

where $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ otherwise. Unlike the traditional Kalman filter, a random bias is also considered to be present. The bias has the assumed properties that

$$\mathrm{E}\left\{\mathbf{b}_\nu\right\} = \mathbf{0}, \qquad \mathrm{E}\left\{\mathbf{b}_\nu\, \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{B}_\nu > \mathbf{O}, \qquad \mathrm{E}\left\{\boldsymbol{\nu}_k \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{O} \;\; \forall k.$$

The shape matrix $\mathbf{\Upsilon}_k$ is deterministic. Since $\boldsymbol{\nu}_k$ and $\mathbf{b}_\nu$ are zero-mean, an unbiased estimation of the state $\widehat{\mathbf{x}}_{k-1}$ can be propagated forward in time to obtain an unbiased estimate at time t_k

$$\widehat{\mathbf{x}}_k^- = \mathbf{\Phi}_k \widehat{\mathbf{x}}_k^+.$$

The estimation error at t_k before the measurement update is defined as

$$\mathbf{e}_k^- \triangleq \mathbf{x}_k - \widehat{\mathbf{x}}_k^-.$$

At t_k, it is assumed that a measurement is available in the form

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{\Lambda}_k \mathbf{b}_\eta + \boldsymbol{\eta}_k,$$

where

$$\begin{aligned} &\mathrm{E}\left\{\boldsymbol{\eta}_k\right\} = \mathbf{0} \;\; \forall\, k, && \mathrm{E}\left\{\boldsymbol{\eta}_j\, \boldsymbol{\eta}_k^{\mathrm{T}}\right\} = \mathbf{R}_k\, \delta_{jk}, && \mathrm{E}\left\{\boldsymbol{\eta}_k \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{O}, \\ &\mathrm{E}\left\{\mathbf{b}_\eta\right\} = \mathbf{0}, && \mathrm{E}\left\{\mathbf{b}_\eta\, \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{B}_\eta > \mathbf{O} && \mathrm{E}\left\{\mathbf{b}_\nu \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{O}, \\ &\mathrm{E}\left\{\boldsymbol{\eta}_j \boldsymbol{\nu}_k\right\} = \mathbf{O}, && \mathrm{E}\left\{\boldsymbol{\nu}_k \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{O} && \mathrm{E}\left\{\mathbf{b}_\eta \boldsymbol{\eta}_k^{\mathrm{T}}\right\} = \mathbf{O}, \end{aligned}$$

for all k, j. The state update is assumed to be the linear update

$$\widehat{\mathbf{x}}_k^+ = \widehat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \widehat{\mathbf{y}}_k), \tag{1}$$

where

$$\widehat{\mathbf{y}}_k \triangleq \mathbf{H}_k \widehat{\mathbf{x}}_k.$$

The update in Eq. (1) provides an unbiased *a posteriori* estimate when the *a priori* estimate is unbiased. After the update, the estimation error is

$$\begin{aligned} \mathbf{e}_k^+ = \mathbf{x}_k - \widehat{\mathbf{x}}_k^+ &= \mathbf{x}_k - \widehat{\mathbf{x}}_k^- - \mathbf{K}_k \left(\mathbf{H}_k \mathbf{x}_k + \boldsymbol{\Lambda}_k \mathbf{b}_\eta + \boldsymbol{\eta}_k - \mathbf{H}_k \widehat{\mathbf{x}}_k^-\right) \\ &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{e}_k^- - \mathbf{K}_k \boldsymbol{\Lambda}_k \mathbf{b}_\eta - \mathbf{K}_k \boldsymbol{\eta}_k. \end{aligned} \tag{2}$$

The covariance update is given by

$$\begin{aligned} \mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k \boldsymbol{\Lambda}_k \mathbf{B}_\eta \boldsymbol{\Lambda}_k^{\mathrm{T}} \mathbf{K}_k^{\mathrm{T}} + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^{\mathrm{T}} + \\ - (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\, \mathrm{E}\left\{\mathbf{e}_k^- \mathbf{b}_\eta^{\mathrm{T}}\right\} \boldsymbol{\Lambda}_k^{\mathrm{T}} \mathbf{K}_k^{\mathrm{T}} - \mathbf{K}_k \boldsymbol{\Lambda}_k\, \mathrm{E}\left\{\mathbf{b}_\eta (\mathbf{e}_k^-)^{\mathrm{T}}\right\} (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^{\mathrm{T}}. \end{aligned} \tag{3}$$

assuming $\boldsymbol{\eta}_k$ and $\mathbf{b}_\eta$ are uncorrelated to the initial estimation error (generally a good assumption). After propagation to the next measurement at time t_{k+1}, the estimation error is

$$\mathbf{e}_{k+1}^- = \mathbf{x}_{k+1} - \widehat{\mathbf{x}}_{k+1}^- = \boldsymbol{\Phi}_k \mathbf{x}_k + \boldsymbol{\Upsilon}_k \mathbf{b}_\nu + \boldsymbol{\nu}_k - \boldsymbol{\Phi}_k \widehat{\mathbf{x}}_k^+ = \boldsymbol{\Phi}_k \mathbf{e}_k^+ + \boldsymbol{\Upsilon}_k \mathbf{b}_\nu + \boldsymbol{\nu}_k. \tag{4}$$

The covariance propagation is given by

$$\mathbf{P}_{k+1}^- = \boldsymbol{\Phi}_k \mathbf{P}_k^+ \boldsymbol{\Phi}_k^{\mathrm{T}} + \boldsymbol{\Upsilon}_k \mathbf{B} \boldsymbol{\Upsilon}_k^{\mathrm{T}} + \mathbf{Q}_k + \boldsymbol{\Phi}_k\, \mathrm{E}\left\{\mathbf{e}_k^+ \mathbf{b}_\nu^{\mathrm{T}}\right\} \boldsymbol{\Upsilon}_k^{\mathrm{T}} + \boldsymbol{\Upsilon}_k\, \mathrm{E}\left\{\mathbf{b}_\nu (\mathbf{e}_k^+)^{\mathrm{T}}\right\} \boldsymbol{\Phi}_k^{\mathrm{T}},$$

assuming $\boldsymbol{\nu}_k$ and $\mathbf{b}_\nu$ are uncorrelated to the initial estimation error (generally a good assumption).

Estimation Error

Substituting Eq. (2) into Eq. (4) yields to the recurrence relation

$$\mathbf{e}_{k+1}^- = \boldsymbol{\Phi}_k \left[(\mathbf{I} - \mathbf{K}_k\, \mathbf{H}_k)\, \mathbf{e}_k^- - \mathbf{K}_k\, \boldsymbol{\eta}_k - \mathbf{K}_k\, \boldsymbol{\Lambda}_k\, \mathbf{b}_\eta\right] + \boldsymbol{\Upsilon}_k \mathbf{b}_\nu + \boldsymbol{\nu}_k.$$

Forming $\mathbf{e}_{k+1}^-\, \mathbf{b}_\eta^{\mathrm{T}}$ and taking the expectation, it follows that

$$\mathrm{E}\left\{\mathbf{e}_{k+1}^-\, \mathbf{b}_\eta^{\mathrm{T}}\right\} = \boldsymbol{\Phi}_k\, (\mathbf{I} - \mathbf{K}_k\, \mathbf{H}_k)\, \mathrm{E}\left\{\mathbf{e}_k^-\, \mathbf{b}_\eta^{\mathrm{T}}\right\} - \boldsymbol{\Phi}_k \mathbf{K}_k\, \boldsymbol{\Lambda}_k \mathbf{B}_\eta. \tag{5}$$

Defining

$$\mathrm{E}\left\{\mathbf{e}_k^-\, \mathbf{b}_\eta^{\mathrm{T}}\right\} \triangleq \mathbf{M}_k\, \mathbf{B}_\eta, \tag{6}$$

and using Eq. (5), the matrix $\mathbf{M}_k$ can be found recursively as

$$\mathbf{M}_{k+1} = \boldsymbol{\Phi}_k \left[(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{M}_k - \mathbf{K}_k \boldsymbol{\Lambda}_k\right].$$

If, at the initial time, a propagation occurs such that

$$\mathbf{e}_1^- = \boldsymbol{\Phi}_0 \mathbf{e}_0 + \boldsymbol{\Upsilon}_0 \mathbf{b}_\nu + \boldsymbol{\nu}_0,$$

then from Eq. (6)

$$\mathrm{E}\left\{\mathbf{e}_1^- \, \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{O} \quad \text{implies that} \quad \mathbf{M}_1 = \mathbf{O}, \text{ since } \mathbf{B}_\eta > \mathbf{O}.$$

Similarly, using Eqs. (2) and (4), it follows that

$$\mathbf{e}_{k+1}^+ = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})\left(\boldsymbol{\Phi}_k\mathbf{e}_k^+ + \boldsymbol{\Upsilon}_k\mathbf{b}_\nu + \boldsymbol{\nu}_k\right) - \mathbf{K}_{k+1}\boldsymbol{\Lambda}_{k+1}\mathbf{b}_\eta - \mathbf{K}_{k+1}\boldsymbol{\eta}_{k+1}.$$

Forming $\mathbf{e}_{k+1}^+ \, \mathbf{b}_\nu^{\mathrm{T}}$ and taking the expectation yields

$$\mathrm{E}\left\{\mathbf{e}_{k+1}^+ \, \mathbf{b}_\nu^{\mathrm{T}}\right\} = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})\left[\boldsymbol{\Phi}_k \, \mathrm{E}\left\{\mathbf{e}_k^+ \, \mathbf{b}_\nu^{\mathrm{T}}\right\} + \boldsymbol{\Upsilon}_k\mathbf{B}_\nu\right].$$

Define

$$\mathrm{E}\left\{\mathbf{e}_k^+ \, \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{L}_k\mathbf{B}_\nu.$$

Then

$$\mathrm{E}\left\{\mathbf{e}_{k+1}^+ \, \mathbf{b}_\nu^{\mathrm{T}}\right\} = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})\left[\boldsymbol{\Phi}_k\mathbf{L}_k + \boldsymbol{\Upsilon}_k\right]\mathbf{B}_\nu = \mathbf{L}_{k+1}\mathbf{B}_\nu,$$

where

$$\mathbf{L}_{k+1} = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})(\boldsymbol{\Phi}_k\mathbf{L}_k + \boldsymbol{\Upsilon}_k). \tag{7}$$

After the first update, we have

$$\mathbf{e}_1^+ = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\left(\boldsymbol{\Phi}_0\mathbf{e}_0 + \boldsymbol{\Upsilon}_0\mathbf{b}_\nu + \boldsymbol{\nu}_0\right) - \mathbf{K}_1\boldsymbol{\Lambda}_1\mathbf{b}_\eta - \mathbf{K}_1\boldsymbol{\eta}_1.$$

Computing $\mathbf{e}_1^+ \, \mathbf{b}_\nu^{\mathrm{T}}$ and taking the expectation yields

$$\mathrm{E}\left\{\mathbf{e}_1^+ \, \mathbf{b}_\nu^{\mathrm{T}}\right\} = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\,\Upsilon_0\mathbf{B}_\nu,$$

since $\mathrm{E}\left\{\mathbf{e}_0\mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{O}$. Therefore, we find that

$$\mathbf{L}_1 = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\boldsymbol{\Upsilon}_0,$$

which can be obtained using the recursion of Eq. (7) for $k = 0$ with $\mathbf{L}_0 = \mathbf{O}$.

Optimal Kalman Gain

Substituting Eq. (6) into Eq. (3), after some rearrangement, we obtain

$$\begin{aligned}\mathbf{P}_k^+ = \mathbf{P}_k^- &- \mathbf{K}_k\big(\mathbf{H}_k\mathbf{P}_k^- + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\big) - \big(\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k\,\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}}\big)\mathbf{K}_k^{\mathrm{T}} + \\ &+ \mathbf{K}_k\big(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}} + \mathbf{H}_k\mathbf{M}_k\,\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}} + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}\big)\mathbf{K}_k^{\mathrm{T}}.\end{aligned}$$

Taking the derivative of the trace of $\mathbf{P}_k^+$ with respect to $\mathbf{K}_k$ yields

$$\begin{aligned}\mathcal{J}' = \frac{d}{d\mathbf{K}_k}\,\mathrm{trace}(\mathbf{P}_k^+) &= -\big(\mathbf{H}_k\mathbf{P}_k^- + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\big)^{\mathrm{T}} - \big(\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k\,\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}}\big) + \\ &+ 2\mathbf{K}_k\big(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}} + \mathbf{H}_k\mathbf{M}_k\,\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}} + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}\big).\end{aligned}$$

Setting $\mathcal{J}' = \mathbf{O}$ and solving for $\mathbf{K}_k$ yields the optimal gain,

$$\mathbf{K}_k = \big(\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k\,\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}}\big)\mathbf{W}_k^{-1}.$$

The matrix $\mathbf{W}_k$ is the covariance of the residuals, as is found to be

$$\begin{aligned}\mathbf{W}_k &\triangleq \mathrm{E}\left\{\boldsymbol{\epsilon}_k\,\boldsymbol{\epsilon}_k^{\mathrm{T}}\right\} = \mathrm{E}\left\{(\mathbf{y} - \widehat{\mathbf{y}})\,(\mathbf{y} - \widehat{\mathbf{y}})^{\mathrm{T}}\right\} = \\ &= \mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}} + \mathbf{H}_k\mathbf{M}_k\,\mathbf{B}_\eta\boldsymbol{\Lambda}_k^{\mathrm{T}} + \boldsymbol{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}.\end{aligned}$$

Notice that when the biases are absent, the filter reduces to the standard Kalman filter. It was assumed that at the initial time a propagation will occur first, and the first update will follow. If an update occurs at time t_0 before the first propagation, the same algorithm can be used by setting

$$\mathbf{M}_0 = \mathbf{O}, \quad \mathbf{L}_0 = \mathbf{O}.$$

MARS ENTRY DYNAMICS MODELING

The dynamic equations in the inertial frame are given by

$$\begin{aligned}
\dot{\mathbf{r}}^i &= \mathbf{v}^i \\
\dot{\mathbf{v}}^i &= \mathbf{g}(\mathbf{r}^i + \mathbf{T}_c^i \mathbf{d}^c) + \mathbf{T}_c^i \mathbf{a}^c \\
\dot{\overline{\mathbf{q}}}_i^c &= \frac{1}{2} \begin{bmatrix} \boldsymbol{\omega}^c \\ 0 \end{bmatrix} \otimes \overline{\mathbf{q}}_i^c .
\end{aligned}$$

The superscript 'c' denotes the IMU case frame. The quaternion is defined with the vector component $\mathbf{q}$ first, and scalar component q last

$$\overline{\mathbf{q}} = [\mathbf{q}^{\mathrm{T}} \;\; q]^{\mathrm{T}}.$$

The quaternion multiplication $\otimes$ is defined such that quaternions multiply in the same order as rotation matrices. All superscripts indicating the inertial or case frame will be dropped since no confusion can arise because each quantity is consistently expressed in the same frame. The vector $\mathbf{r}$ is the position of the IMU in the inertial frame, $\mathbf{v}$ is the velocity of the IMU in the inertial frame, $\overline{\mathbf{q}}$ is the quaternion expressing the rotation from inertial to case, therefore the rotation matrix from the inertial frame to the case frame is $\mathbf{T} \triangleq \mathbf{T}_i^c = \mathbf{T}(\overline{\mathbf{q}})$. The vector $\mathbf{g}$ is the acceleration due to gravity, $\mathbf{d}$ is the offset between the IMU and the center of mass which is expressed in the case frame. The vector $\mathbf{a}$ is the true non-gravitational acceleration represented in the IMU case frame, and $\boldsymbol{\omega}$ is the relative angular velocity vector of the IMU case frame with respect to the inertial frame expressed in the case frame.

SENSOR ERROR MODELS

Only the strapdown implementation of the IMU is considered. The IMU's accelerometers and gyros produce measurements corrupted by random errors (noise and biases), systematic biases, and other errors. The simplified dynamics model used for dead-reckoning navigation, minimizes the contribution of process noise. The two main sources of error are the IMU errors and the accuracy of the initial state estimate.

Accelerometers

The accelerometer package produces a measure of the spacecraft change in velocity due to non-gravitational accelerations in the IMU case frame, denoted by $\Delta\mathbf{v}$. This measurement is corrupted by errors due to nonorthogonality and misalignment of the axes, errors due to scale factor uncertainties, random biases, and noise. The accelerometer error model can be formulated as

$$\begin{aligned}
\Delta\mathbf{v}_{true,k} &= \int_{t_{k-1}}^{t_k} \mathbf{a} \; dt \\
\Delta\mathbf{v}_k &= (\mathbf{I}_{3\times 3} + \boldsymbol{\Gamma}_a)(\mathbf{I}_{3\times 3} + \mathbf{S}_a)(\Delta\mathbf{v}_{true,k} + \mathbf{b}_a + \boldsymbol{\xi}_k),
\end{aligned} \tag{8}$$

where

$$\boldsymbol{\Gamma}_a \triangleq \begin{bmatrix} 0 & \gamma_{a_{xz}} & -\gamma_{a_{xy}} \\ -\gamma_{a_{yz}} & 0 & \gamma_{a_{yx}} \\ \gamma_{a_{zy}} & -\gamma_{a_{zx}} & 0 \end{bmatrix}, \qquad \mathbf{S}_a \triangleq \begin{bmatrix} s_{a_x} & 0 & 0 \\ 0 & s_{a_y} & 0 \\ 0 & 0 & s_{a_z} \end{bmatrix},$$

and $(\gamma_{a_{yz}}, \gamma_{a_{zy}}, \gamma_{a_{zx}}, \gamma_{a_{xz}}, \gamma_{a_{xy}}, \gamma_{a_{yx}})$ are nonorthogonality and axes misalignment errors, $\mathbf{b}_a \in \Re^3$ is the bias in the accelerometer, $(s_{a_x}, s_{a_y}, s_{a_z})$ are scale factor errors, and $\boldsymbol{\xi}_k \in \Re^3$ is a white sequence stochastic process. The nonorthogonality and axes misalignment errors, scale factor errors, and bias parameters are all modeled as zero-mean random constants. The noise $\boldsymbol{\xi}_k$ is modeled as zero-mean, random sequence. If we assume that the various errors are "small," then to first-order we have

$$(\mathbf{I}_{3\times 3} + \boldsymbol{\Gamma}_a)(\mathbf{I}_{3\times 3} + \mathbf{S}_a) \approx \mathbf{I}_{3\times 3} + \boldsymbol{\Gamma}_a + \mathbf{S}_a.$$

Defining

$$\boldsymbol{\Delta}_a \triangleq \boldsymbol{\Gamma}_a + \mathbf{S}_a \tag{9}$$

yields the accelerometer measurement model

$$\Delta \mathbf{v}_k = (\mathbf{I} + \boldsymbol{\Delta}_a)(\Delta \mathbf{v}_{true,k} + \mathbf{b}_a + \boldsymbol{\xi}_k). \tag{10}$$

Gyros

The gyro package produces a measure of the spacecraft attitude change, denoted by $\Delta\boldsymbol{\theta}_k$. The measurement of the angular velocity vector is corrupted by random biases, errors due to scale factor uncertainties, errors due to nonorthogonality and axes misalignments, and random noise. The gyro error model can be formulated as

$$\begin{aligned} \Delta\boldsymbol{\theta}_{true,k} &= \int_{t_{k-1}}^{t_k} \boldsymbol{\omega}\ dt \\ \Delta\boldsymbol{\theta}_k &= (\mathbf{I}_{3\times3} + \boldsymbol{\Gamma}_g)(\mathbf{I}_{3\times3} + \mathbf{S}_g)(\Delta\boldsymbol{\theta}_{true,k} + \mathbf{b}_g + \boldsymbol{\eta}_k)\ , \end{aligned} \tag{11}$$

where $\mathbf{b}_g$ is the gyro bias, $\mathbf{S}_g$ is the gyro scale factor matrix, $\boldsymbol{\Gamma}_g$ is the gyro nonorthogonality and axes misalignment matrix, $\boldsymbol{\eta}_k$ is a white sequence, and where

$$\mathbf{S}_g \triangleq \begin{bmatrix} s_{g_x} & 0 & 0 \\ 0 & s_{g_y} & 0 \\ 0 & 0 & s_{g_z} \end{bmatrix}, \qquad \boldsymbol{\Gamma}_g \triangleq \begin{bmatrix} 0 & \gamma_{g_{xz}} & -\gamma_{g_{xy}} \\ -\gamma_{g_{yz}} & 0 & \gamma_{g_{yx}} \\ \gamma_{g_{zy}} & -\gamma_{g_{zx}} & 0 \end{bmatrix}, \tag{12}$$

and $(\gamma_{g_{yz}}, \gamma_{g_{zy}}, \gamma_{g_{zx}}, \gamma_{g_{xz}}, \gamma_{g_{xy}}, \gamma_{g_{yx}})$ are nonorthogonality and axes misalignment errors, $\mathbf{b}_g \in \Re^3$ is the bias in the gyro, $(s_{g_x}, s_{g_y}, s_{g_z})$ are scale factor errors, and $\boldsymbol{\eta}_k \in \Re^3$ is a white sequence. The nonorthogonality and axes misalignment errors, scale factor errors, and bias parameters are all modeled as zero-mean random constants. The noise $\boldsymbol{\eta}_k$ is modeled as a zero-mean, white random sequence. To first-order, we have

$$(\mathbf{I}_{3\times3} + \mathbf{S}_g)(\mathbf{I}_{3\times3} + \boldsymbol{\Gamma}_g) \approx \mathbf{I}_{3\times3} + \mathbf{S}_g + \boldsymbol{\Gamma}_g\ .$$

Hence, Eq. (11) can be written in the form

$$\Delta\boldsymbol{\theta}_k = (\mathbf{I}_{3\times3} + \boldsymbol{\Delta}_g)(\Delta\boldsymbol{\theta}_{true,k} + \mathbf{b}_g + \boldsymbol{\eta}_k), \tag{13}$$

where $\boldsymbol{\Delta}_g \triangleq \mathbf{S}_g + \boldsymbol{\Gamma}_g$.

PROPAGATION

Suppose that we have available IMU measurements of $\Delta\mathbf{v}$'s and $\Delta\boldsymbol{\theta}$'s. The standard procedure is to assume $\mathbf{a}$ and $\boldsymbol{\omega}$ constant over the time step, t_k - t_{k-1}, such that

$$\mathbf{a}_k \triangleq \frac{\Delta\mathbf{v}_k}{\Delta t}, \qquad \boldsymbol{\omega}_k \triangleq \frac{\Delta\boldsymbol{\theta}_k}{\Delta t}, \qquad \forall t \in [t_{k-1}, t_k].$$

The quaternion expressing the rotation from inertial to case at time t_{k-1} is denoted by $\overline{\mathbf{q}}_{k-1}$. Define the quaternion $\Delta\overline{\mathbf{q}}$ expressing the rotation during one time step as,

$$\Delta\widehat{\overline{\mathbf{q}}}(t) \triangleq \widehat{\overline{\mathbf{q}}}(t) \otimes \widehat{\overline{\mathbf{q}}}_{k-1}^{-1}, \qquad t \in [t_{k-1},\, t_k].$$

Its evolution is given by

$$\Delta\dot{\widehat{\overline{\mathbf{q}}}}(t) = \dot{\widehat{\overline{\mathbf{q}}}}(t) \otimes \widehat{\overline{\mathbf{q}}}_{k-1}^{-1} = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_k)\Delta\overline{\mathbf{q}}(t), \qquad t \in [t_{k-1},\, t_k]. \tag{14}$$

Let $\boldsymbol{\theta}$ be the rotation vector parametrization of $\Delta\overline{\mathbf{q}}$, using this parametrization and assuming small $\boldsymbol{\theta}$ (i.e. small time step), Eq. (14) becomes

$$\dot{\widehat{\boldsymbol{\theta}}}(t) = \boldsymbol{\omega}_k - \boldsymbol{\omega}_k \times \widehat{\boldsymbol{\theta}}(t), \qquad t \in [t_{k-1},\, t_k], \qquad \widehat{\boldsymbol{\theta}}(t_{k-1}) = \mathbf{0}. \tag{15}$$

The solution of Eq. (15) is

$$\widehat{\boldsymbol{\theta}}(t) = \boldsymbol{\omega}_k (t - t_{k-1}),$$

therefore the discrete quaternion update is given by

$$\widehat{\overline{\mathbf{q}}}_k = \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k) \otimes \widehat{\overline{\mathbf{q}}}_{k-1},$$

where

$$\overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k) = \begin{bmatrix} \sin\left(\frac{1}{2}\|\Delta\boldsymbol{\theta}_k\|\right) \cdot \Delta\boldsymbol{\theta}_k / \|\Delta\boldsymbol{\theta}_k\| \\ \cos(\frac{1}{2}\|\Delta\boldsymbol{\theta}_k\|) \end{bmatrix}.$$

the estimate of the velocity evolves as

$$\dot{\widehat{\mathbf{v}}}(t) = \mathbf{g}(\widehat{\mathbf{r}} + \widehat{\mathbf{T}}(t)^{\mathrm{T}}\widehat{\mathbf{d}}) + \widehat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{a}_k, \qquad t \in [t_{k-1},\, t_k], \tag{16}$$

The estimate of the rotation matrix is

$$\widehat{\mathbf{T}}(t)^{\mathrm{T}} = \mathbf{T}\big(\widehat{\overline{\mathbf{q}}}(t)\big)^{\mathrm{T}} = \mathbf{T}(\widehat{\overline{\mathbf{q}}}_{k-1})^{\mathrm{T}}\, \mathbf{T}(\Delta\widehat{\overline{\mathbf{q}}}(t)\,)^{\mathrm{T}}, \qquad t \in [t_{k-1}, t_k],$$

to first order

$$\Delta\mathbf{T}(t) \triangleq \mathbf{T}(\Delta\widehat{\overline{\mathbf{q}}}(t)\,) \simeq \mathbf{I}_{3\times 3} - [\widehat{\boldsymbol{\theta}}(t)\times], \qquad t \in [t_{k-1}, t_k].$$

We assume that

$$\begin{aligned} \widehat{\mathbf{g}}(t) &\triangleq \mathbf{g}(\widehat{\mathbf{r}}(t) + \widehat{\mathbf{T}}(t)^{\mathrm{T}}\widehat{\mathbf{d}}) \simeq \mathbf{g}\big(\widehat{\mathbf{r}}_{k-1} + \widehat{\mathbf{T}}(t)^{\mathrm{T}}\widehat{\mathbf{d}}\big) \\ &\simeq \mathbf{g}\big(\widehat{\mathbf{r}}_{k-1} + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\widehat{\mathbf{d}}\big) + \left.\frac{\partial \mathbf{g}(\mathbf{r})}{\partial \mathbf{r}}\right|_{\widehat{\mathbf{r}}_{k-1}+\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\widehat{\mathbf{d}}} \Big(\widehat{\mathbf{T}}(t)^{\mathrm{T}} - \widehat{\mathbf{T}}_{k-1}\Big)\,\widehat{\mathbf{d}} \\ &= \widehat{\mathbf{g}}_{k-1} + \widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\boldsymbol{\theta}}(t)\times]\widehat{\mathbf{d}}, \qquad \forall\, t \in [t_{k-1}, t_k], \end{aligned}$$

which implies that we are assuming the acceleration of gravity at the IMU location is constant during the time step. The contribution of $\widehat{\mathbf{G}}_{k-1}[\widehat{\boldsymbol{\theta}}(t)\times]\widehat{\mathbf{d}}$ is negligible but will be kept for completeness. Eq. (16) becomes

$$\begin{aligned} \dot{\widehat{\mathbf{v}}}(t) &= \widehat{\mathbf{g}}_{k-1} + \widehat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{a}_k = \widehat{\mathbf{g}}_{k-1} - \widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\,\widehat{\boldsymbol{\theta}}(t) + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\Delta\widehat{\mathbf{T}}(t)\mathbf{a}_k \\ &\simeq \widehat{\mathbf{g}}_{k-1} + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\mathbf{a}_k - \Big(\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times] + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\mathbf{a}_k\times]\Big)\,\widehat{\boldsymbol{\theta}}(t), \qquad t \in [t_{k-1}, t_k], \end{aligned} \tag{17}$$

integrating Eq. (17) we obtain

$$\widehat{\mathbf{v}}_k = \widehat{\mathbf{v}}_{k-1} + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\Delta\mathbf{v}_k + \widehat{\mathbf{g}}_{k-1}\Delta t - \frac{1}{2}\Big(\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times] + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\mathbf{a}_k\times]\Big)\,\Delta\boldsymbol{\theta}_k\Delta t,$$

integrating Eq. (17) twice yields

$$\widehat{\mathbf{r}}_k = \widehat{\mathbf{r}}_{k-1} + \widehat{\mathbf{v}}_{k-1}\Delta t + \frac{1}{2}\widehat{\mathbf{g}}_{k-1}\Delta t^2 + \frac{1}{2}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\Delta\mathbf{v}_k\Delta t - \frac{1}{6}\Big(\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times] + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\mathbf{a}_k\times]\Big)\,\Delta\boldsymbol{\theta}_k\Delta t^2.$$

In summary the estimated states are obtained solving

$$\begin{aligned} \widehat{\mathbf{r}}_k &= \widehat{\mathbf{r}}_{k-1} + \widehat{\mathbf{v}}_{k-1}\Delta t + \frac{1}{2}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times 3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right)\Delta\mathbf{v}_k\Delta t + \frac{1}{2}\left(\widehat{\mathbf{g}}_{k-1} - \frac{1}{3}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\,\Delta\boldsymbol{\theta}_k\right)\Delta t^2 \\ \widehat{\mathbf{v}}_k &= \widehat{\mathbf{v}}_{k-1} + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times 3} + \frac{1}{2}\,[\Delta\boldsymbol{\theta}_k\times]\right)\Delta\mathbf{v}_k + \left(\widehat{\mathbf{g}}_{k-1} - \frac{1}{2}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\Delta\boldsymbol{\theta}_k\right)\Delta t \\ \widehat{\overline{\mathbf{q}}}_k &= \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k) \otimes \widehat{\overline{\mathbf{q}}}_{k-1}. \end{aligned} \tag{18}$$

If it is desired to have a more accurate representation of the gravitational acceleration, the time step can be divided to use a higher order method, each sub-step will employ an equation similar to Eq. (18), then all contribution will be added together in a weighted average. Notice that only the contribution due to gravity will be represented more accurately. Relying solely on the IMU integral measurements, point-wise in time quantities are not available and discretization errors are unavoidable. This approach would be preferable if the IMU was providing measurements at a low frequency; normally the IMU can function at ten Hertz or higher, which makes the assumption of constant gravitational acceleration in between measurements very reasonable.

Solution of the navigation equations given in Eq. (18) yields the navigated spacecraft position, velocity, and attitude. The navigation system, also provides a measure of the estimated states uncertainty. The accuracy of the navigated state depends strongly on knowledge of the initial spacecraft state. Also, any measurement errors present in $\Delta\boldsymbol{\theta}_k$ and $\Delta\mathbf{v}_k$ will corrupt the navigation solution.

As can be seen in Eq. (18), solution of the position and velocity equations requires knowledge of the attitude to rotate the IMU accelerations from the case frame to the inertial frame. Therefore the estimation error of position and velocity is coupled to the attitude error. The attitude estimation does not rely on the position and velocity estimation, hence can be addressed independently.

Attitude estimation errors

It is assumed that

$$\overline{\mathbf{q}}_k = \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_{true,k}) \otimes \overline{\mathbf{q}}_{k-1}. \tag{19}$$

This is only an approximation, but the discretization error will be compensated via process noise. Define the multiplicative attitude error as

$$\delta\overline{\mathbf{q}} \triangleq \overline{\mathbf{q}} \otimes \widehat{\overline{\mathbf{q}}}^{-1}.$$

Using Eqs. (18) and (19) yields

$$\begin{aligned}
\delta\overline{\mathbf{q}}_k &= \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_{true,k}) \otimes \overline{\mathbf{q}}_{k-1} \otimes \widehat{\overline{\mathbf{q}}}_{k-1}^{-1} \otimes \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k)^{-1} \\
&= \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_{true,k}) \otimes \delta\overline{\mathbf{q}}_{k-1} \otimes \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k)^{-1} \\
&= \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_{true,k}) \otimes \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k)^{-1} \otimes \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k) \otimes \delta\overline{\mathbf{q}}_{k-1} \otimes \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k)^{-1},
\end{aligned}$$

which can be re-written as

$$\delta\overline{\mathbf{q}}_k = \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_{true,k}) \otimes \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k)^{-1} \otimes \left(\begin{bmatrix} \mathbf{T}(\Delta\boldsymbol{\theta}_k) & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \delta\overline{\mathbf{q}}_{k-1} \right). \tag{20}$$

Assuming small angles, the vector component of the quaternion fully represents the attitude

$$\delta\overline{\mathbf{q}} \simeq \begin{bmatrix} \delta\mathbf{q} \\ 1 \end{bmatrix}, \qquad \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_{true,k}) \otimes \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k)^{-1} \simeq \begin{bmatrix} \frac{1}{2}\left(\Delta\boldsymbol{\theta}_{true,k} - \Delta\boldsymbol{\theta}_k\right) \\ 1 \end{bmatrix}.$$

Employing Eq. (20) and approximating to first-order yields

$$\delta\mathbf{q}_k = \mathbf{T}(\Delta\boldsymbol{\theta}_k)\,\delta\mathbf{q}_{k-1} + \frac{1}{2}\left(\Delta\boldsymbol{\theta}_{true,k} - \Delta\boldsymbol{\theta}_k\right).$$

It then follows from Eq. (13) that

$$\delta\mathbf{q}_k = \mathbf{T}(\Delta\boldsymbol{\theta}_k)\,\delta\mathbf{q}_{k-1} + \frac{1}{2}\left[(\mathbf{I} + \boldsymbol{\Delta}_g)^{-1}\Delta\boldsymbol{\theta}_k - \mathbf{b}_g - \boldsymbol{\xi}_k - \Delta\boldsymbol{\theta}_k\right].$$

To first-order in $\boldsymbol{\Delta}_g$ we have

$$(\mathbf{I} + \boldsymbol{\Delta}_g)^{-1} \simeq \mathbf{I} - \boldsymbol{\Delta}_g.$$

It then follows that (to first-order)

$$\delta \mathbf{q}_k = \mathbf{T}(\Delta\boldsymbol{\theta}_k)\,\delta\mathbf{q}_{k-1} - \frac{1}{2}\left(\boldsymbol{\Delta}_g \Delta\boldsymbol{\theta}_k + \mathbf{b}_g + \boldsymbol{\xi}_k\right) \ .$$

With the given definition of $\mathbf{S}_g$ in Eq. (12), we can write

$$\mathbf{S}_g \Delta\boldsymbol{\theta}_k = \mathbf{D}(\Delta\boldsymbol{\theta}_k)\mathbf{s}_g,$$

where

$$\mathbf{s}_g \triangleq \begin{bmatrix} s_{gx} \\ s_{gy} \\ s_{gz} \end{bmatrix} \quad \text{and} \quad \mathbf{D}(\Delta\boldsymbol{\theta}_k) \triangleq \begin{bmatrix} \Delta\theta_{k_x} & 0 & 0 \\ 0 & \Delta\theta_{k_y} & 0 \\ 0 & 0 & \Delta\theta_{k_z} \end{bmatrix} . \tag{21}$$

Similarly, we can write

$$\boldsymbol{\Gamma}_g \Delta\boldsymbol{\theta}_k = \mathbf{N}(\Delta\boldsymbol{\theta}_k)\boldsymbol{\gamma}_g \ , \tag{22}$$

where

$$\boldsymbol{\gamma}_g \triangleq \begin{bmatrix} \gamma_{g_{xy}} \\ \gamma_{g_{xz}} \\ \gamma_{g_{yx}} \\ \gamma_{g_{yz}} \\ \gamma_{g_{zx}} \\ \gamma_{g_{zy}} \end{bmatrix} \quad \text{and} \quad \mathbf{N}(\Delta\boldsymbol{\theta}_k) \triangleq \begin{bmatrix} -\Delta\theta_{k_z} & \Delta\theta_{k_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta\theta_{k_z} & -\Delta\theta_{k_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta\theta_{k_y} & \Delta\theta_{k_x} \end{bmatrix} .$$

For small angles the rotation vector $\boldsymbol{\theta}$ is approximately twice the vector part of the quaternion, therefore it follows that the estimation error represented with the rotation vector is given by

$$\boldsymbol{e}_{\theta,k} = \mathbf{T}(\Delta\boldsymbol{\theta}_k)\,\boldsymbol{e}_{\theta,k-1} - \mathbf{D}(\Delta\boldsymbol{\theta}_k)\mathbf{s}_g - \mathbf{N}(\Delta\boldsymbol{\theta}_k)\boldsymbol{\gamma}_g - \mathbf{b}_g - \boldsymbol{\xi}_k. \tag{23}$$

Position and velocity estimation errors

The assumption made in Eq. (19) is equivalent to assuming constant angular velocity in between measurements. Similarly the gravitational and nongravitational accelerations will be assumed constant during the time step. These assumptions lead to equations for the propagation of the true state equivalent to Eq. (18) yields

$$\begin{aligned}
\mathbf{r}_k &= \mathbf{r}_{k-1} + \mathbf{v}_{k-1}\Delta t + \frac{1}{2}\mathbf{g}_{k-1}\Delta t^2 + \frac{1}{2}\mathbf{T}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times3} + \frac{1}{3}\left[\Delta\boldsymbol{\theta}_{true,k}\times\right]\right)\Delta\mathbf{v}_{true,k}\Delta t + \\
&\quad - \frac{1}{6}\mathbf{G}_{k-1}\mathbf{T}_{k-1}^{\mathrm{T}}[\mathbf{d}\times]\Delta\boldsymbol{\theta}_{true,k}\Delta t^2. \\
\mathbf{v}_k &= \mathbf{v}_{k-1} + \mathbf{g}_{k-1}\Delta t + \mathbf{T}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times3} + \frac{1}{2}\left[\Delta\boldsymbol{\theta}_{true,k}\times\right]\right)\Delta\mathbf{v}_{true,k} - \frac{1}{2}\mathbf{G}_{k-1}\mathbf{T}_{k-1}^{\mathrm{T}}[\mathbf{d}\times]\Delta\boldsymbol{\theta}_{true,k}\Delta t.
\end{aligned}$$

To compensate for the error introduced by the discretization, process noise will be added in the Kalman filter. The position and velocity estimation error are defined to be

$$\mathbf{e}_r \triangleq \mathbf{r} - \widehat{\mathbf{r}} \qquad \text{and} \qquad \mathbf{e}_v \triangleq \mathbf{v} - \widehat{\mathbf{v}}.$$

Computing $\mathbf{e}_r$ yields

$$\begin{aligned}
\mathbf{e}_{r,k} &= \mathbf{e}_{r,k-1} + \mathbf{e}_{v,k-1}\Delta t + \frac{1}{2}(\mathbf{g}_{k-1} - \widehat{\mathbf{g}}_{k-1})\Delta t^2 - \frac{1}{6}\mathbf{G}_{k-1}\mathbf{T}_{k-1}^{\mathrm{T}}[\mathbf{d}\times]\Delta\boldsymbol{\theta}_{true,k}\Delta t^2 + \\
&\quad + \frac{1}{6}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\Delta\boldsymbol{\theta}_k\Delta t^2 + \frac{1}{2}\mathbf{T}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times3} + \frac{1}{3}\left[\Delta\boldsymbol{\theta}_{true,k}\times\right]\right)\left(\Delta\mathbf{v}_{true,k}\right)\Delta t + \\
&\quad - \frac{1}{2}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times3} + \frac{1}{3}\left[\Delta\boldsymbol{\theta}_k\times\right]\right)\Delta\mathbf{v}_k\Delta t,
\end{aligned}$$

vector $\widehat{\mathbf{d}}$ is the estimate of the distance between the IMU and the center of mass. Expanding gravity, utilizing a Taylor series and neglecting higher order terms, it follows that

$$\mathbf{g}(\mathbf{r}+\mathbf{T}^{\mathrm{T}}\mathbf{d})-\mathbf{g}(\widehat{\mathbf{r}}+\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{d}})\simeq\widehat{\mathbf{G}}\left(\mathbf{e}_r+\mathbf{T}^{\mathrm{T}}\mathbf{d}-\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{d}}\right),$$

where

$$\widehat{\mathbf{G}}\triangleq\left.\frac{\partial\mathbf{g}}{\partial\mathbf{r}}\right|_{\mathbf{r}=\widehat{\mathbf{r}}+\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{d}}}.$$

Since the quaternion error is defined as $\delta\overline{\mathbf{q}}\triangleq\overline{\mathbf{q}}\otimes\widehat{\overline{\mathbf{q}}}^{-1}$ and attitude matrices are multiplied in the same order as quaternions, then $\delta\mathbf{T}=\mathbf{T}\,\widehat{\mathbf{T}}^{\mathrm{T}}$. Therefore,

$$\mathbf{T}^{\mathrm{T}}\mathbf{d}-\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{d}}=\widehat{\mathbf{T}}^{\mathrm{T}}\,\delta\mathbf{T}^{\mathrm{T}}(\widehat{\mathbf{d}}+\mathbf{e}_d)-\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{d}},$$

where $\mathbf{e}_d\triangleq\mathbf{d}-\widehat{\mathbf{d}}$. To first-order it follows that

$$\delta\mathbf{T}^{\mathrm{T}}\simeq\mathbf{I}_{3\times3}+[\mathbf{e}_\theta\times].$$

Then

$$\mathbf{T}^{\mathrm{T}}\mathbf{d}-\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{d}}\simeq\widehat{\mathbf{T}}^{\mathrm{T}}[\mathbf{e}_\theta\times]\widehat{\mathbf{d}}+\widehat{\mathbf{T}}^{\mathrm{T}}\mathbf{e}_d=-\widehat{\mathbf{T}}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\mathbf{e}_\theta+\widehat{\mathbf{T}}^{\mathrm{T}}\,\mathbf{e}_d.$$

Similarly

$$\mathbf{T}^{\mathrm{T}}\Delta\mathbf{v}_{true}-\widehat{\mathbf{T}}^{\mathrm{T}}\Delta\mathbf{v}=\widehat{\mathbf{T}}^{\mathrm{T}}\,\delta\mathbf{T}^{\mathrm{T}}\Delta\mathbf{v}_{true}-\widehat{\mathbf{T}}^{\mathrm{T}}\Delta\mathbf{v}.$$

Hence, to first-order, we have

$$\mathbf{T}^{\mathrm{T}}\Delta\mathbf{v}_{true}-\widehat{\mathbf{T}}^{\mathrm{T}}\Delta\mathbf{v}\simeq\widehat{\mathbf{T}}^{\mathrm{T}}[\mathbf{e}_\theta\times]\Delta\mathbf{v}+\widehat{\mathbf{T}}^{\mathrm{T}}\left(\Delta\mathbf{v}_{true}-\Delta\mathbf{v}\right).$$

Finally, the position estimation error is obtained to first-order as

$$\begin{aligned}\mathbf{e}_{r,k}=\;&\mathbf{e}_{r,k-1}+\mathbf{e}_{v,k-1}\Delta t-\frac{1}{2}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left[\left(\mathbf{I}_{3\times3}+\frac{1}{3}\left[\Delta\boldsymbol{\theta}_{m,k}\times\right]\right)\Delta\mathbf{v}_{m,k}\times\right]\mathbf{e}_\theta\Delta t\\&+\frac{1}{2}\widehat{\mathbf{G}}_{k-1}\left(\mathbf{e}_r+\frac{1}{3}\widehat{\mathbf{T}}^{\mathrm{T}}[(\widehat{\mathbf{d}}\times\Delta\boldsymbol{\theta}_{m,k})\times]\mathbf{e}_\theta+\frac{1}{3}\widehat{\mathbf{T}}^{\mathrm{T}}\left[\Delta\boldsymbol{\theta}_{m,k}\times\right]\mathbf{e}_d\right)\Delta t^2+\\&+\frac{1}{2}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times3}+\frac{1}{3}\left[\Delta\boldsymbol{\theta}_{m,k}\times\right]\right)\left(\Delta\mathbf{v}_{true,k}-\Delta\mathbf{v}_{m,k}\right)\Delta t-\frac{1}{6}\widehat{\mathbf{U}}_{k-1}\mathbf{e}_r\Delta t^2+\\&-\frac{1}{6}\left(\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\Delta\mathbf{v}_{m,k}\times]+\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\Delta t\right)(\Delta\boldsymbol{\theta}_{true,k}-\Delta\boldsymbol{\theta}_{m,k})\Delta t.\end{aligned}\tag{24}$$

Following a similar pattern, the velocity estimation error is given by

$$\begin{aligned}\mathbf{e}_{v,k}=\;&\mathbf{e}_{v,k-1}+\widehat{\mathbf{G}}_{k-1}\mathbf{e}_r\Delta t-\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left[\left(\mathbf{I}_{3\times3}+\frac{1}{2}\left[\Delta\boldsymbol{\theta}_{m,k}\times\right]\right)\Delta\mathbf{v}_{m,k}\times\right]\mathbf{e}_\theta\\&+\frac{1}{2}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}\left(\left[\left(\widehat{\mathbf{d}}\times\Delta\boldsymbol{\theta}_{m,k}\right)\times\right]\mathbf{e}_\theta+[\Delta\boldsymbol{\theta}_{m,k}\times]\mathbf{e}_d\right)\Delta t+\\&+\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times3}+\frac{1}{2}\left[\Delta\boldsymbol{\theta}_{m,k}\times\right]\right)\left(\Delta\mathbf{v}_{true,k}-\Delta\mathbf{v}_{m,k}\right)-\frac{1}{2}\widehat{\mathbf{U}}_{k-1}\mathbf{e}_r\Delta t+\\&-\frac{1}{2}\left(\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\Delta\mathbf{v}_{m,k}\times]+\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\Delta t\right)(\Delta\boldsymbol{\theta}_{true,k}-\Delta\boldsymbol{\theta}_{m,k}).\end{aligned}\tag{25}$$

In Eqs. (24) and (25), the ij-th component of matrix $\widehat{\mathbf{U}}$ is defined as

$$\widehat{U}(ij)\triangleq\left.\sum_{l=1}^{3}\frac{\partial^2 g(i)}{\partial r(j)\,\partial r(l)}\,u(l)\right|_{\mathbf{r}=\widehat{\mathbf{r}}+\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{d}}}$$

$$\mathbf{u}\triangleq\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\Delta\boldsymbol{\theta}_{m,k}.$$

This term arises from the difference in gravitational acceleration between the center of mass and the IMU location, and can be neglected in most practical applications. Rearranging the terms in the accelerometer model given in Eq. (10) yields

$$\Delta\mathbf{v} = (\mathbf{I} + \boldsymbol{\Delta}_a)^{-1}\Delta\mathbf{v}_{true} - (\mathbf{b}_a + \boldsymbol{\xi})$$

where, after some manipulation and using the fact that for "small" $\boldsymbol{\Delta}_a$ we have $(\mathbf{I}+\boldsymbol{\Delta}_a)^{-1} \simeq \mathbf{I}-\boldsymbol{\Delta}_a$, we obtain

$$\Delta\mathbf{v}_{true} - \Delta\mathbf{v} = -\boldsymbol{\Delta}_a\Delta\mathbf{v} - (\mathbf{b}_a + \boldsymbol{\xi}).$$

The matrix $\boldsymbol{\Delta}_a$ is comprised of random constants, $\mathbf{b}_a$ is a random constant vector, $\mathbf{e}_d$ is a random constant if we assume ballistic entry, and $\boldsymbol{\xi}$ is a random sequence. Consider the term $\boldsymbol{\Delta}_a\Delta\mathbf{v}$ more closely. From the definition of $\boldsymbol{\Delta}_a$ given in Eq. (9) we have

$$\boldsymbol{\Delta}_a\Delta\mathbf{v} = (\boldsymbol{\Gamma}_a + \mathbf{S}_a)\,\Delta\mathbf{v}.$$

With the definitions of $\boldsymbol{\Gamma}_a$ and $\mathbf{S}_a$ given in Eq. (8), we find that we can also write $\boldsymbol{\Delta}_a\mathbf{a}_k$ as

$$\boldsymbol{\Delta}_a\Delta\mathbf{v} = \mathbf{D}(\Delta\mathbf{v})\mathbf{s}_a + \mathbf{N}(\Delta\mathbf{v})\boldsymbol{\gamma}_a,$$

where definitions equivalent to those of Eqs. (21) and (22) are used.

Collecting the position, velocity, and attitude estimation error equations from Eqs. (23)–(25), and writing in matrix form yields the stochastic linear matrix difference equation

$$\mathbf{e}_k = \boldsymbol{\Phi}_{k-1}\mathbf{e}_{k-1} + \boldsymbol{\Upsilon}_{k-1}\mathbf{b}_\nu + \mathbf{J}_{k-1}\boldsymbol{\nu}_{k-1}, \tag{26}$$

where

$$\mathbf{e}_k \triangleq \begin{bmatrix} \mathbf{e}_{r,k} \\ \mathbf{e}_{v,k} \\ \mathbf{e}_{\theta,k} \end{bmatrix} \in \Re^9, \qquad \mathbf{b}_\nu \triangleq \begin{bmatrix} \mathbf{s}_a \\ \boldsymbol{\gamma}_a \\ \mathbf{b}_a \\ \mathbf{s}_g \\ \boldsymbol{\gamma}_g \\ \mathbf{b}_g \\ \mathbf{e}_d \end{bmatrix} \in \Re^{27}, \qquad \boldsymbol{\nu}_{k-1} \triangleq \begin{bmatrix} \boldsymbol{\xi}_k \\ \boldsymbol{\eta}_k \end{bmatrix} \in \Re^6.$$

The error state matrix $\boldsymbol{\Phi}_k \in \Re^{9\times 9}$ is

$$\boldsymbol{\Phi}_{k-1} = \begin{bmatrix} \mathbf{I}_{3\times3} + \frac{1}{2}\widehat{\mathbf{G}}_{k-1}\Delta t^2 & \mathbf{I}_{3\times3}\Delta t & \frac{1}{2}\left\{\frac{1}{3}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}[(\widehat{\mathbf{d}}\times\Delta\boldsymbol{\theta}_k)\times]\Delta t - \widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left[\left(\mathbf{I}_{3\times3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right)\Delta\mathbf{v}_k\times\right]\right\}\Delta t \\ \widehat{\mathbf{G}}_{k-1}\Delta t & \mathbf{I}_{3\times3} & \frac{1}{2}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}\left[\left(\widehat{\mathbf{d}}\times\Delta\boldsymbol{\theta}_k\right)\times\right]\Delta t - \widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left[\left(\mathbf{I}_{3\times3} + \frac{1}{2}[\Delta\boldsymbol{\theta}_k\times]\right)\Delta\mathbf{v}_k\times\right] \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{T}(\Delta\boldsymbol{\theta}_k) \end{bmatrix},$$

and the input mapping matrices are $\boldsymbol{\Upsilon}_k \in \Re^{9\times 27}$ and $\mathbf{J}_k \in \Re^{9\times 6}$ are

$$\boldsymbol{\Upsilon}_{k-1} = \begin{bmatrix}\boldsymbol{\Upsilon}_{a,k-1} & \boldsymbol{\Upsilon}_{g,k-1} & \boldsymbol{\Upsilon}_{d,k-1}\end{bmatrix},$$

$$\boldsymbol{\Upsilon}_{a,k-1} = \begin{bmatrix} -\frac{1}{2}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left(\mathbf{I}_{3\times3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right)\begin{bmatrix}\mathbf{D}(\Delta\mathbf{v}_k) & \mathbf{N}(\Delta\mathbf{v}_k) & \mathbf{I}_{3\times3}\end{bmatrix}\Delta t \\ -\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left(\mathbf{I}_{3\times3} + \frac{1}{2}[\Delta\boldsymbol{\theta}_k\times]\right)\begin{bmatrix}\mathbf{D}(\Delta\mathbf{v}_k) & \mathbf{N}(\Delta\mathbf{v}_k) & \mathbf{I}_{3\times3}\end{bmatrix} \\ \mathbf{O}_{3\times9} \end{bmatrix}$$

$$\boldsymbol{\Upsilon}_{g,k-1} = \begin{bmatrix} \frac{1}{6}\left(\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\Delta\mathbf{v}_k\times] + \widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\widehat{\mathbf{d}}\times]\Delta t\right)\begin{bmatrix}\mathbf{D}(\Delta\boldsymbol{\theta}_k) & \mathbf{N}(\Delta\boldsymbol{\theta}_k) & \mathbf{I}_{3\times3}\end{bmatrix}\Delta t \\ \frac{1}{2}\left(\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\Delta\mathbf{v}_k\times] + \widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\widehat{\mathbf{d}}\times]\Delta t\right)\begin{bmatrix}\mathbf{D}(\Delta\boldsymbol{\theta}_k) & \mathbf{N}(\Delta\boldsymbol{\theta}_k) & \mathbf{I}_{3\times3}\end{bmatrix} \\ -\begin{bmatrix}\mathbf{D}(\Delta\boldsymbol{\theta}_k) & \mathbf{N}(\Delta\boldsymbol{\theta}_k) & \mathbf{I}_{3\times3}\end{bmatrix} \end{bmatrix}$$

$$\boldsymbol{\Upsilon}_{d,k-1} = \begin{bmatrix} \frac{1}{6}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}[\Delta\boldsymbol{\theta}_k\times]\Delta t^2 \\ \frac{1}{2}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}[\Delta\boldsymbol{\theta}_k\times]\Delta t \\ \mathbf{O}_{3\times3} \end{bmatrix}$$

and

$$\mathbf{J}_{k-1} = \begin{bmatrix} -\frac{1}{2}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times 3} + \frac{1}{3}\left[\Delta\boldsymbol{\theta}_k\times\right]\right)\Delta t & \frac{1}{6}\left(\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\Delta\mathbf{v}_k\times] + \widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\Delta t\right)\Delta t \\ -\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times 3} + \frac{1}{2}\left[\Delta\boldsymbol{\theta}_k\times\right]\right) & \frac{1}{2}\left(\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\Delta\mathbf{v}_k\times] + \widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}[\widehat{\mathbf{d}}\times]\Delta t\right) \\ \mathbf{O}_{3\times 3} & -\mathbf{I}_{3\times 3} \end{bmatrix}.$$

The components of $\mathbf{b}_\nu$ in Eq. (26) are the various random constant errors associated with the IMU, where it assumed that

$$\mathrm{E}\left\{\mathbf{b}_\nu\right\} = \mathbf{0}$$

and $\mathbf{B}_\nu \in \Re^{27\times 27}$ is

$$\mathbf{B}_\nu \triangleq \mathrm{E}\left\{\mathbf{b}_\nu\mathbf{b}_\nu^{\mathrm{T}}\right\}.$$

The components of $\boldsymbol{\nu}_k$ in Eq. (26) are the non-constant random components of the IMU errors, where it is assumed that

$$\mathrm{E}\left\{\boldsymbol{\nu}_k\right\} = \mathbf{0} \qquad \text{and} \qquad \mathrm{E}\left\{\boldsymbol{\nu}_i\boldsymbol{\nu}_j^{\mathrm{T}}\right\} = \mathbf{V}_i\delta_{ij}.$$

It is now left to calculate the covariance evolution from Eq. (26). One way is to augment the state vector with the bias $\mathbf{b}_\nu$. The solution chosen here is different: the bias will not be included as a state variable, but directly as a source of error, following the approach previously developed.

Estimation error covariance

If no updates occur, the error covariance in the IMU dead-reckoning case can be computed with the previously derived equation

$$\mathbf{P}_k^- = \boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1}^+\boldsymbol{\Phi}_{k-1}^{\mathrm{T}} + \boldsymbol{\Upsilon}_{k-1}\mathbf{B}_\nu\boldsymbol{\Upsilon}_{k-1}^{\mathrm{T}} + \mathbf{Q}_{k-1} + \boldsymbol{\Phi}_{k-1}\mathbf{L}_{k-1}\mathbf{B}_\nu\boldsymbol{\Upsilon}_{k-1}^{\mathrm{T}} + \boldsymbol{\Upsilon}_{k-1}\mathbf{B}_\nu\mathbf{L}_{k-1}^{\mathrm{T}}\boldsymbol{\Phi}_{k-1}^{\mathrm{T}}$$

where

$$\mathbf{L}_k = \boldsymbol{\Phi}_{k-1}\mathbf{L}_{k-1} + \boldsymbol{\Upsilon}_{k-1}, \qquad \mathbf{L}_0 = \mathbf{O},$$

and

$$\mathbf{Q}_k = \mathbf{J}_k\mathbf{V}_k\mathbf{J}_k^{\mathrm{T}}.$$

DEAD RECKONING NAVIGATION

Suppose that the IMU observations, $\Delta\mathbf{v}_k$ and $\Delta\boldsymbol{\theta}_k$ are available. Then, dead reckoning navigation, including computing the associated state estimation error covariance, is the process of the following equations at each time t_k a measurement is available:

$$\begin{aligned}
\widehat{\mathbf{r}}_k &= \widehat{\mathbf{r}}_{k-1} + \widehat{\mathbf{v}}_{k-1}\Delta t + \frac{1}{2}\widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times 3} + \frac{1}{3}\left[\Delta\boldsymbol{\theta}_k\times\right]\right)\Delta\mathbf{v}_k\Delta t + \left(\frac{1}{2}\widehat{\mathbf{g}}_{k-1} - \frac{1}{6}\widehat{\mathbf{G}}_{k-1}[\widehat{\mathbf{d}}\times]\,\Delta\boldsymbol{\theta}_k\right)\Delta t^2 \\
\widehat{\mathbf{v}}_k &= \widehat{\mathbf{v}}_{k-1} + \widehat{\mathbf{T}}_{k-1}^{\mathrm{T}}\left(\mathbf{I}_{3\times 3} + \frac{1}{2}\left[\Delta\boldsymbol{\theta}_k\times\right]\right)\Delta\mathbf{v}_k + \left(\widehat{\mathbf{g}}_{k-1} - \frac{1}{2}\widehat{\mathbf{G}}_{k-1}[\widehat{\mathbf{d}}\times]\Delta\boldsymbol{\theta}_k\right)\Delta t \\
\widehat{\overline{\mathbf{q}}}_k &= \overline{\mathbf{q}}(\Delta\boldsymbol{\theta}_k)\otimes\widehat{\overline{\mathbf{q}}}_{k-1} \\
\mathbf{L}_k &= \mathbf{F}_{k-1}\mathbf{L}_{k-1} + \mathbf{H}_{k-1} \\
\mathbf{P}_k &= \mathbf{F}_{k-1}\mathbf{P}_{k-1}\mathbf{F}_{k-1}^{\mathrm{T}} + \mathbf{J}_{k-1}\mathbf{V}_{k-1}\mathbf{J}_{k-1}^{\mathrm{T}} + \mathbf{H}_{k-1}\mathbf{W}\mathbf{H}_{k-1}^{\mathrm{T}} + \mathbf{F}_{k-1}\mathbf{L}_{k-1}\mathbf{W}\mathbf{H}_{k-1}^{\mathrm{T}} + \mathbf{H}_{k-1}\mathbf{W}\mathbf{L}_{k-1}^{\mathrm{T}}\mathbf{F}_{k-1}^{\mathrm{T}}.
\end{aligned}$$

where $\widehat{\mathbf{g}}_k \triangleq \mathbf{g}(\widehat{\mathbf{r}}_k + \widehat{\mathbf{T}}_k^{\mathrm{T}}\widehat{\mathbf{d}})$ is the modeled gravity, $\widehat{\overline{\mathbf{q}}} = \begin{bmatrix}\widehat{\mathbf{q}}^{\mathrm{T}} & \widehat{q}\end{bmatrix}^{\mathrm{T}}$ and

$$\widehat{\mathbf{T}}_k^{\mathrm{T}} \triangleq \mathbf{T}(\widehat{\overline{\mathbf{q}}})^{\mathrm{T}} = \mathbf{I}_{3\times 3} + 2\widehat{q}_k[\widehat{\mathbf{q}}_k\times] + 2[\widehat{\mathbf{q}}_k\times]^2,$$

$$\Phi_k \triangleq \begin{bmatrix} \mathbf{I}_{3\times3} + \frac{1}{2}\widehat{\mathbf{G}}_{k-1}\Delta t^2 & \mathbf{I}_{3\times3}\Delta t & \frac{1}{2}\left\{\frac{1}{3}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}[(\widehat{\mathbf{d}} \times \Delta\boldsymbol{\theta}_k)\times]\Delta t - \widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left[\left(\mathbf{I}_{3\times3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right)\Delta\mathbf{v}_k\times\right]\right\}\Delta t \\ \widehat{\mathbf{G}}_{k-1}\Delta t & \mathbf{I}_{3\times3} & \frac{1}{2}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}\left[\left(\widehat{\mathbf{d}} \times \Delta\boldsymbol{\theta}_k\right)\times\right]\Delta t - \widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left[\left(\mathbf{I}_{3\times3} + \frac{1}{2}[\Delta\boldsymbol{\theta}_k\times]\right)\Delta\mathbf{v}_k\times\right] \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{T}(\Delta\boldsymbol{\theta}_k) \end{bmatrix},$$

$$\boldsymbol{\Upsilon}_{k-1} \triangleq \begin{bmatrix} -\frac{1}{2}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left(\mathbf{I}_{3\times3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right)\left[\mathbf{D}(\Delta\mathbf{v}_k) \quad \mathbf{N}(\Delta\mathbf{v}_k) \quad \mathbf{I}_{3\times3}\right]\Delta t & \frac{1}{6}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\Delta\mathbf{v}_k\times]\left[\mathbf{D}(\Delta\boldsymbol{\theta}_k) \quad \mathbf{N}(\Delta\boldsymbol{\theta}_k) \quad \mathbf{I}_{3\times3}\right]\Delta t & \frac{1}{2}\widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}\Delta t^2 \\ -\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left(\mathbf{I}_{3\times3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right)\left[\mathbf{D}(\Delta\mathbf{v}_k) \quad \mathbf{N}(\Delta\mathbf{v}_k) \quad \mathbf{I}_{3\times3}\right] & \frac{1}{2}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\Delta\mathbf{v}_k\times]\left[\mathbf{D}(\Delta\boldsymbol{\theta}_k) \quad \mathbf{N}(\Delta\boldsymbol{\theta}_k) \quad \mathbf{I}_{3\times3}\right] & \widehat{\mathbf{G}}_{k-1}\widehat{\mathbf{T}}^{\mathrm{T}}\Delta t \\ \mathbf{O}_{3\times9} & -\left[\mathbf{D}(\Delta\boldsymbol{\theta}_k) \quad \mathbf{N}(\Delta\boldsymbol{\theta}_k) \quad \mathbf{I}_{3\times3}\right] & \mathbf{O}_{3\times3} \end{bmatrix},$$

$$\mathbf{J}_{k-1} \triangleq \begin{bmatrix} -\frac{1}{2}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left(\mathbf{I}_{3\times3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right)\Delta t & \frac{1}{6}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\Delta\mathbf{v}_k\times]\Delta t \\ -\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}\left(\mathbf{I}_{3\times3} + \frac{1}{3}[\Delta\boldsymbol{\theta}_k\times]\right) & \frac{1}{2}\widehat{\mathbf{T}}^{\mathrm{T}}_{k-1}[\Delta\mathbf{v}_k\times] \\ \mathbf{O}_{3\times3} & -\mathbf{I}_{3\times3} \end{bmatrix},$$

$$\mathbf{D}(\Delta\mathbf{v}) \triangleq \begin{bmatrix} \Delta v_x & 0 & 0 \\ 0 & \Delta v_y & 0 \\ 0 & 0 & \Delta v_z \end{bmatrix}, \qquad \mathbf{N}(\Delta\mathbf{v}_m) \triangleq \begin{bmatrix} -\Delta v_z & \Delta v_y & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta v_z & -\Delta v_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta v_y & \Delta v_x \end{bmatrix},$$

$$\mathbf{D}(\Delta\boldsymbol{\theta}) \triangleq \begin{bmatrix} \Delta\theta_x & 0 & 0 \\ 0 & \Delta\theta_y & 0 \\ 0 & 0 & \Delta\theta_z \end{bmatrix}, \qquad \mathbf{N}(\Delta\boldsymbol{\theta}) \triangleq \begin{bmatrix} -\Delta\theta_z & \Delta\theta_y & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta\theta_z & -\Delta\theta_x & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta\theta_y & \Delta\theta_x \end{bmatrix},$$

with initial conditions

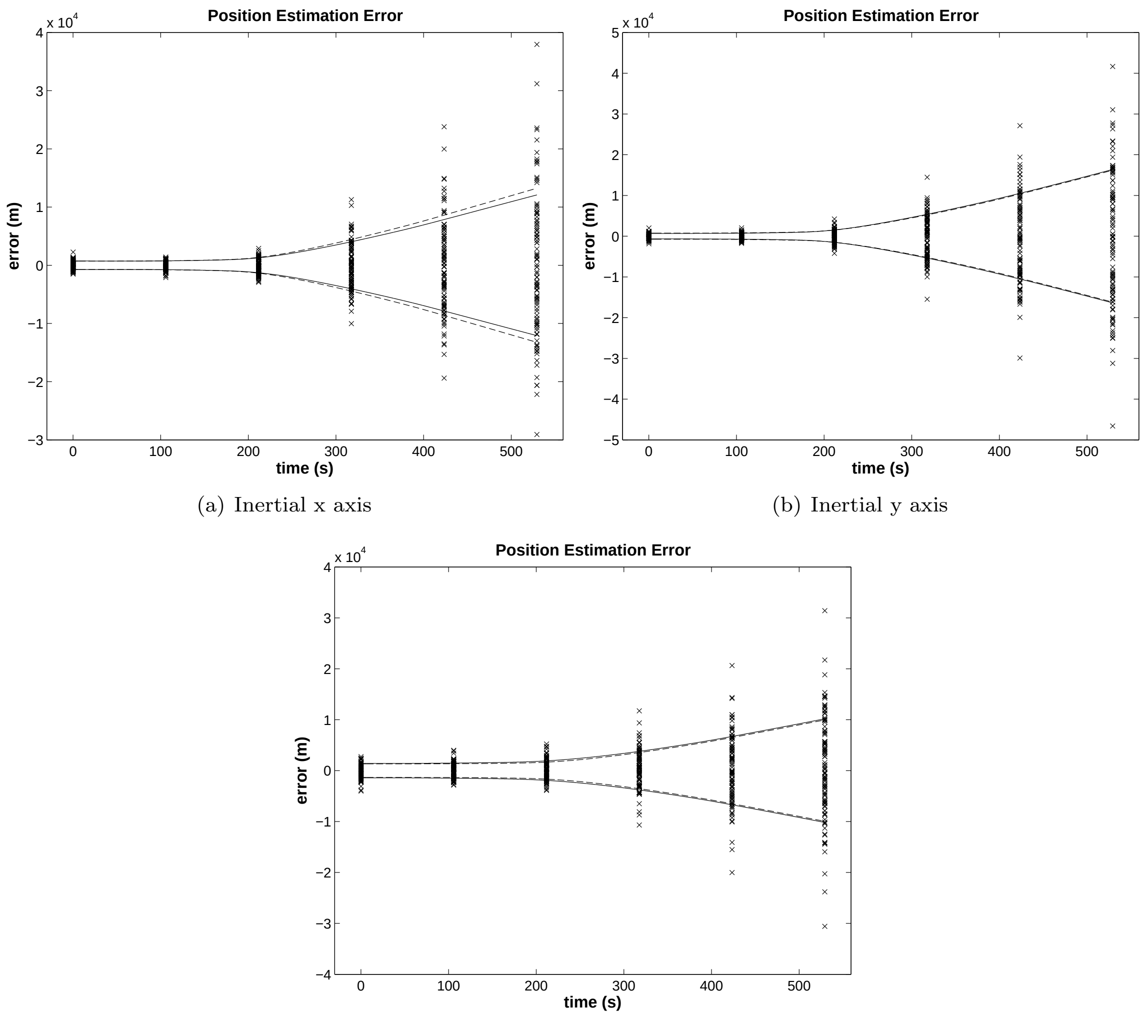
$$\widehat{\mathbf{r}}_0 = \widehat{\mathbf{r}}(t_0), \quad \widehat{\mathbf{v}}_0 = \widehat{\mathbf{v}}(t_0), \quad \widehat{\bar{\mathbf{q}}}_0 = \widehat{\bar{\mathbf{q}}}(t_0), \quad \mathbf{P}_0 = \mathbf{P}(t_0), \quad \mathbf{L}_0 = \mathbf{0}.$$

The sensor provide discrete measurements $\Delta\mathbf{v}_k$ and $\Delta\boldsymbol{\theta}_k$, and their error models are assumed known *apriori* and represented by the matrices $\mathbf{V}_k$ and $\mathbf{B}_\nu$.

SIMULATION RESULTS

The linear covariance formulation thus described has been tested using a trajectory generated by NASA JSC SORT simulation program. SORT is a high fidelity simulation used to provide the true states and true measurements. Verification of the formulation is made through use of Monte Carlo analysis corrupting the measurements and the initial estimate. The IMU measurements are corrupted with all the sources previously described. By explicitly accounting for all the errors, the uncertainty due to the random biased can be quantified in a single run, which aids the filter tuning process. When biases are not directly accounted for, their contribution is introduced by adjusting the filter covariances. Monte Carlo analysis becomes necessary to show that the covariance obtained through the “tuning” matches the statistical covariance.

In the section, we compare the sampled estimation error covariance obtained through Monte Carlo analysis with the linear covariance formulation to demonstrate the performance of the linear covariance formulation. Figures 1–3 show samples of error evolution in each of the 100 runs (denoted by “x”), the sample covariance (continuous lines), and the linear covariance formulation evaluated (dashed lines). Figures 1–2 contain the inertial position and velocity errors in the x, y, and z axis respectively. Figure 3 contains the three components of the attitude error from estimated body frame to true body frame, the attitude error is represented as a rotation vector.

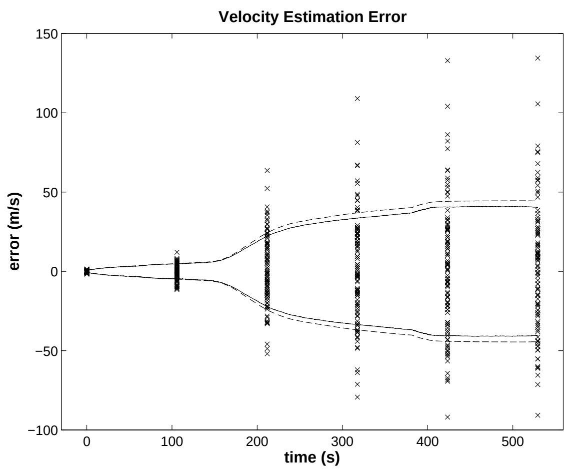


(a) Inertial x axis

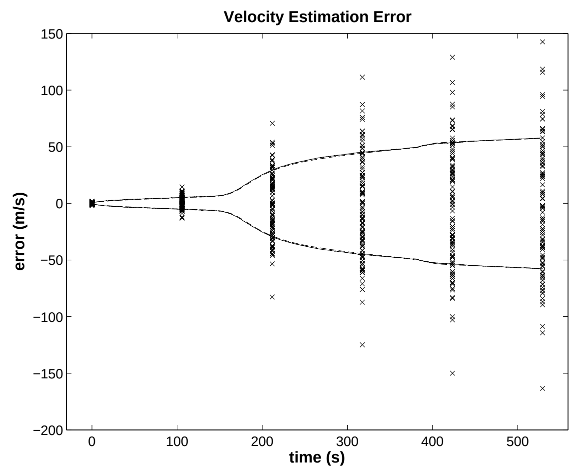
(b) Inertial y axis

(c) Inertial z axis

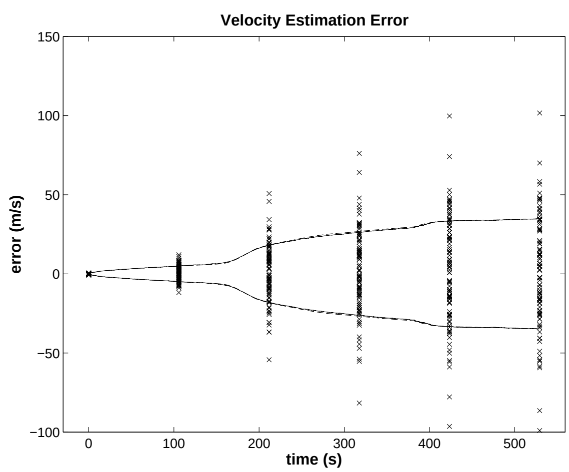
Figure 1: Position estimation error. Error denoted by "x", sample covariance by continuous line, and calculated covariance by dashed line.



(a) Inertial x axis

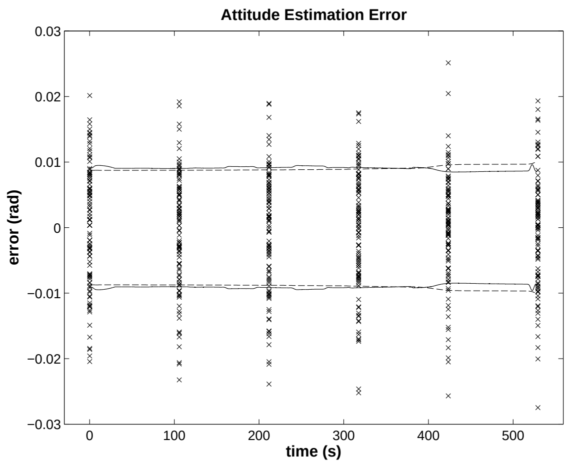


(b) Inertial y axis

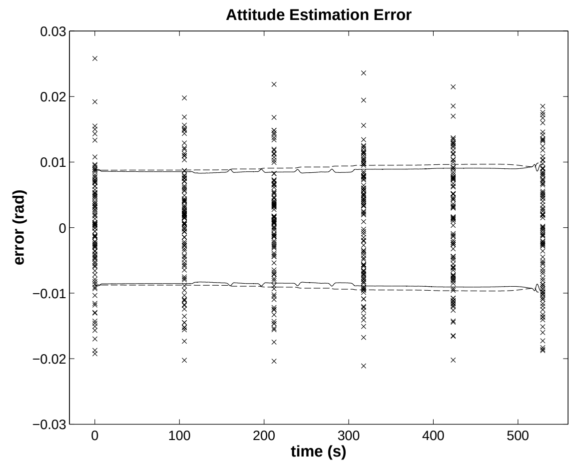


(c) Inertial z axis

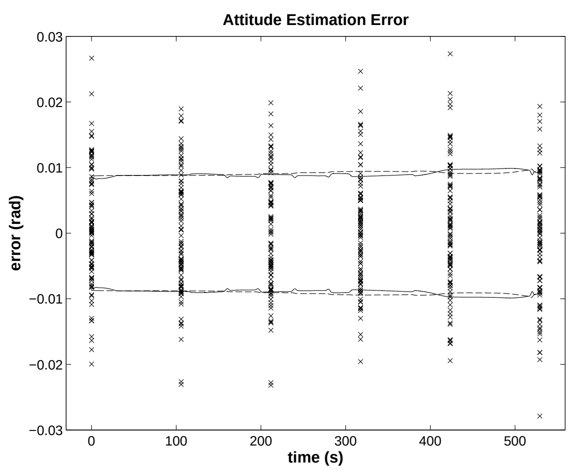
Figure 2: Velocity estimation error. Error denoted by "x", sample covariance by continuous line, and calculated covariance by dashed line.



(a) First component



(b) Second component



(c) Third component

Figure 3: Attitude estimation error. Error denoted by “x”, sample covariance by continuous line, and calculated covariance by dashed line.

CONCLUSIONS

In this work, the algorithms for precise dead-reckoning navigation were derived to include the state estimation error covariance computation. The underlying error equations were linearized and utilized to develop a formulation of the approximate state estimation error covariance. The correlation of attitude errors with position and velocity errors was explicitly derived. The resulting set of dead-reckoning relationships can be used as an independent verification of Monte Carlo analysis during the verification of the entry filter.

Although no external measurements were simulated, the presented algorithm allows for their easy introduction to update the state.

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