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RELATIVE NAVIGATION FOR THE ORION VEHICLE

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The Orion vehicle is being designed to provide manned spaceflight capability after the retirement of the Space Shuttle in 2010. Orion will provide access to both the International Space Station and the Moon. In both cases, the vehicle is required to perform rendezvous and therefore requires a relative navigation filter. This paper documents the preliminary analysis performed by the Orion Navigation team, and reviews the current Orion relative navigation architecture as well as the sensors available for rendezvous and proximity operations.

INTRODUCTION

Orion will perform a rendezvous in low Earth orbit (LEO) during missions to the International Space Station (ISS) and the Moon. It will rendezvous with the Space Station and with the Earth Departure Stage/Lunar Surface Access Module (LSAM) on those missions, respectively. In addition LSAM will perform a rendezvous with Orion in low lunar orbit (LLO). The preliminary design of the relative navigation algorithm to be used by Orion during Rendezvous, Proximity Operations, and Docking (RPOD) is detailed in this paper. The purpose of the algorithm is the determination of relative position and velocity between Orion and its rendezvous target during RPOD. The filter is only operational during RPOD and is initialized from the ground and the absolute navigation algorithm prior to the initiation of the rendezvous.

Orion's absolute navigation system will be used for some of the Fault Detection, Isolation, and Recovery (FDIR) functions required by the relative navigation algorithm, as well as for interfacing with the various IMU, GPS, and star tracker boxes. In addition to measurements from absolute sensors, the relative filter processes measurements from three types of relative sensors: star trackers (target bearing), the communications system (radiometric ranging and bearing), and Vision Navigation Sensors (VNS) (laser-based range, bearing, and pose).

The relative navigation filter maintains two inertial states, each composed of position and velocity: one for Orion and one for the target vehicle. An estimate of attitude is also maintained in the filter, as are various sensor misalignment and bias states. An estimate of Orion's angular rate is not kept in the filter because the rates from the gyros are used to integrate the attitude state. In the event that sensor data is lost, the filter can continue to propagate the vehicle states via dead-reckoning as long as the IMU data is available from the absolute navigation algorithm, which is not detailed in this paper. Propagation can continue in any case without IMU data, but would only be of use while in coasting flight. Any relative state parameters are calculated in the User Parameter Processor (UPP) as necessary for guidance and control.

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It is anticipated that the relative filter will receive incremental changes in sensed velocity and attitude of the vehicle at 40 Hz from the IMU selected by the absolute navigation system. These values will be accumulated by the relative filter at 40 Hz and used for propagation at the slower relative filter execution rate. Each sensor that provides measurements to the filter has a sensor level FDIR algorithm, which feeds the selection filter for that sensor. Only the VNS and RF tracker FDIR and selection filter algorithms fall under the relative navigation domain. During the portion of the rendezvous trajectory during which Orion is processing relative measurements, the relative navigation filter will have the ability to process relative measurements from multiple types of sensors at once (but not from multiple sensors of the same type). Sensor misalignment and bias states will be initialized and maintained in the filter corresponding to the selected sensors. After the state and covariance have been updated by the filter based on the available measurements, the UPP algorithm updates the state that it uses to calculate the relative navigation parameters that are used by guidance and control. The UPP algorithm propagates this state at a high rate determined by the needs of guidance and control, but does not propagate the covariance. Additional capabilities include the ability to incorporate state updates from the ground or the absolute navigation algorithm, either as a re-initialization or a measurement, and the maintenance of a propagated target state which is used as a backup in the case that bad sensor data corrupts the filter state.

The Orion attitude states are handled in the manner of a multiplicative extended Kalman filter (MEKF) [1]. The purpose of maintaining an estimate of Orion's attitude in the relative filter is to capture the error introduced into relative bearing measurements due to attitude knowledge uncertainty. The MEKF formulation that is used to maintain the Orion attitude states is incorporated into the main filter, as opposed to being a separate attitude filter. The attitude states represent a small rotation from a reference attitude quaternion to the estimated attitude of Orion, in the form of scaled Modified Rodrigues Parameters (MRP). The attitude deviation states are used to update the reference quaternion in a multiplicative fashion after measurement processing, and are subsequently zeroed out. Thus, only the reference quaternion is propagated from one time step to the next in the filter.

FILTER STATE DESCRIPTION

State Vector Definition

The elements that make up the state vector during RPOD are listed in Table 1, though the filter state will only contain the sensor misalignment and bias states when the corresponding sensors are active. The Orion position and velocity states represent the absolute inertial position and velocity of the selected IMU on the Orion vehicle, as opposed to the vehicle c.g. The reason for this convention is to avoid integrating gyro noise into the position estimate, which occurs when the sensed $\Delta \mathbf{v}$ from the IMU must be compensated for the lever arm effect between the IMU box and the vehicle c.g. in order to calculate the sensed $\Delta \mathbf{v}$ of the c.g.

When the filter is just propagating the states of the two vehicles and differencing the result (before relative measurements are available), the filter does not contain the sensor error states. However, error states are added to the filter state vector as measurements become available. When a bearing measurement becomes available, be it from a star tracker, VNS, or the communications system, the bearing sensor misalignment and bias states, $\boldsymbol{\gamma}$ and $\mathbf{b}_{angles}$, are initialized with values that correspond to the chosen bearing sensor. When range is provided by either the communications system or the VNS, the range measurement bias is initialized (and the range rate measurement bias

Table 1 State Element Definitions

States	Description
$\mathbf{r}^i_{orion}$	Orion absolute position vector, referenced to selected IMU location
$\dot{\mathbf{r}}^i_{orion}$	Orion absolute velocity vector, referenced to selected IMU location
$\mathbf{p}_b^{b_{ref}}$	Orion estimated body to reference body attitude rotation, represented as Modified Rodrigues Parameters scaled by 4
$\mathbf{r}^i_{targ}$	Target absolute position vector, referenced to c.g.
$\dot{\mathbf{r}}^i_{targ}$	Target absolute velocity vector, referenced to c.g.
$\boldsymbol{\gamma}$	Bearing sensor misalignment rotation vector, defining actual sensor frame wrt reference sensor frame
$\mathbf{b}_\alpha$	A 2×1 Vector of bearing measurement biases
b_ρ	Range measurement bias
$b_{\dot{\rho}}$	Range rate measurement bias

in the case of the communications system measurements) in a similar manner. The filter state vector at its maximum extent contains 22 elements, and is given by

$$\mathbf{X} = \left[\begin{array}{cccccccccc} \mathbf{r}^{i\mathrm{T}}_{orion} & \dot{\mathbf{r}}^{i\mathrm{T}}_{orion} & \left(\mathbf{p}_b^{b_{ref}}\right)^{\mathrm{T}} & \mathbf{r}^{i\mathrm{T}}_{targ} & \dot{\mathbf{r}}^{i\mathrm{T}}_{targ} & \boldsymbol{\gamma}^{\mathrm{T}} & \mathbf{b}^{\mathrm{T}}_{angles} & b_\rho & b_{\dot{\rho}} \end{array} \right]^{\mathrm{T}} \tag{1}$$

Attitude Conventions

Orion's reference attitude is stored in a body-to-inertial left quaternion, denoted

$$\bar{\mathbf{q}}^i_{b_{ref}} = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} \tag{2}$$

The quaternion multiplication convention followed herein is the original definition by Hamilton and is denoted by "$\circledast$"

$$\bar{\mathbf{q}} \circledast \bar{\mathbf{l}} = \begin{bmatrix} q_0 l_0 - \mathbf{q} \cdot \mathbf{l} \\ q_0 \mathbf{l} + l_0 \mathbf{q} + \mathbf{q} \times \mathbf{l} \end{bmatrix} = \begin{bmatrix} q_0 & -\mathbf{q}^{\mathrm{T}} \\ \mathbf{q} & (q_0 \mathbf{I}_{3\times3} + [\mathbf{q}\times]) \end{bmatrix} \begin{bmatrix} l_0 \\ \mathbf{l} \end{bmatrix} \tag{3}$$

where

$$[\mathbf{q}\times] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{4}$$

The reference body to inertial transformation matrix can be obtained, in terms of the reference quaternion, from the quaternion rotation operator,

$$\bar{\mathbf{q}}^i_{b_{ref}} \begin{bmatrix} 0 \\ \mathbf{v}^b \end{bmatrix} \left(\bar{\mathbf{q}}^i_{b_{ref}}\right)^* = \begin{bmatrix} 0 \\ \mathbf{T}^i_{b_{ref}} \mathbf{v}^b \end{bmatrix} \tag{5}$$

where $\left(\bar{\mathbf{q}}^i_{b_{ref}}\right)^*$ is the conjugate of $\bar{\mathbf{q}}^i_{b_{ref}}$ (the vector part of the quaternion is multiplied by -1), and $\mathbf{v}^b$ is an arbitrary vector in the Orion body frame. The result is

$$\mathbf{T}^i_{b_{ref}} = \mathbf{I}_{3\times3} + 2q_0\left[\mathbf{q}\times\right] + 2\left[\mathbf{q}\times\right]^2 \tag{6}$$

The actual filter states that are used to update the estimate of Orion's attitude are a set of scaled MRPs, which represent a deviation from the reference attitude. Modified Rodriguez Parameters were chosen for the attitude representation in the filter for two reasons: 1) the three parameters are independent (in contrast to a quaternion, which contains four parameters subject to one normalization constraint, so there is one dependent parameter), and 2) the MRP do not require trigonometric function evaluations to generate transformation matrices as do rotation vectors. Given the deviation from the actual body to the reference body frame in the form of a unit quaternion, $\delta\bar{\mathbf{q}}^{b_{ref}}_b = \begin{bmatrix} \delta q_0 & \delta\mathbf{q}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, the scaled MRP can be calculated as

$$\mathbf{p}^{b_{ref}}_b = \frac{4\delta\mathbf{q}}{1+\delta q_0} \tag{7}$$

Note that a MRP is a 2:1 attitude mapping, because $\delta\bar{\mathbf{q}}$ and $-\delta\bar{\mathbf{q}}$ represent the same rotation, but yield different MRP's. Therefore, it is required in the filter that $\delta q_0 \geq 0$. If $\delta q_0 < 0$, then the quaternion must be negated prior to the computation of a scaled MRP in the filter. This policy also prevents Eq. (7) from becoming singular.

The actual body-to-inertial attitude quaternion can be calculated from $\bar{\mathbf{q}}^i_b = \bar{\mathbf{q}}^i_{b_{ref}} \circledast \delta\bar{\mathbf{q}}^{b_{ref}}_b$, The deviation quaternion is calculated from the scaled MRP by

$$\delta\bar{\mathbf{q}}^{b_{ref}}_b = \begin{bmatrix} \frac{16-\mathbf{p}^{\mathrm{T}}\mathbf{p}}{16+\mathbf{p}^{\mathrm{T}}\mathbf{p}} \\ \left(\frac{8}{16+\mathbf{p}^{\mathrm{T}}\mathbf{p}}\right)\mathbf{p} \end{bmatrix} \tag{8}$$

where $\mathbf{p} \equiv \mathbf{p}^{b_{ref}}_b$. For small rotations, $\mathbf{p} \approx -\boldsymbol{\phi}$, where $\boldsymbol{\phi}$ is the Euler angle/axis defining the single axis rotation from the actual body frame to the reference body frame. The actual body to reference body transformation matrix can be represented exactly in terms of $\boldsymbol{\phi}$ as [3]

$$\mathbf{T}^{b_{ref}}_b = \frac{\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}}{\phi^2}\left(1-\cos\phi\right) + \mathbf{I}_{3\times3}\cos\phi - \frac{\sin\phi}{\phi}\left[\boldsymbol{\phi}\times\right] \tag{9}$$

where $\phi = \|\boldsymbol{\phi}\|$ is the magnitude of the rotation vector.

Additionally, the transformation matrix, $\mathbf{T}^{b_{ref}}_b$, can be approximated in terms of the scaled MRP to second order accuracy by writing it in terms of $\mathbf{p}$ through the use of Eqs. (6) and (8), noting that $\frac{1}{(16+\mathbf{p}^{\mathrm{T}}\mathbf{p})^2} \approx \frac{1}{16^2} - \frac{2}{16^3}\mathbf{p}^{\mathrm{T}}\mathbf{p}$ to second order when expanded about $\mathbf{p} = \mathbf{0}$, and neglecting terms higher than second order in the resulting equation:

$$\begin{aligned} \mathbf{T}^{b_{ref}}_b &= \frac{1}{\left(16+\mathbf{p}^{\mathrm{T}}\mathbf{p}\right)^2}\left(\left[\left(16+\mathbf{p}^{\mathrm{T}}\mathbf{p}\right)^2 - 64\mathbf{p}^{\mathrm{T}}\mathbf{p}\right]\mathbf{I}_{3\times3} + 128\mathbf{p}\mathbf{p}^{\mathrm{T}} + 16\left(16-\mathbf{p}^{\mathrm{T}}\mathbf{p}\right)\left[\mathbf{p}\times\right]\right) \\ &\approx \mathbf{I}_{3\times3} + \left[\mathbf{p}\times\right] - \frac{1}{2}\left(\mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I}_{3\times3} - \mathbf{p}\mathbf{p}^{\mathrm{T}}\right) \end{aligned} \tag{10}$$

where the result differs slightly from that of Landis Markley due to the use of left quaternions.

STATE AND COVARIANCE PROPAGATION

Filter propagation will occur at a rate of 5 Hz or less, however, the absolute navigation algorithm will output incremental changes in attitude and sensed velocity from its IMU compensation algorithm at a rate of either 50 Hz or 200 Hz. The relative navigation algorithm thus has an IMU accumulator function that runs at the higher rate of the IMU compensation algorithm and accumulates the incremental changes in attitude and sensed velocity for use by the slower relative navigation propagation function.

Actual relative navigation filter propagation, which occurs at a rate not exceeding 5 Hz, consists of three components: 1) transformation of sensed $\Delta\mathbf{v}$ into inertial frame and reference quaternion update, 2) filter state propagation, and 3) state transition matrix propagation/covariance matrix update.

It is anticipated that a 8×8 gravity model will be used for propagation during RPOD in LEO, while at least a 16×16 model will be used in LLO. Partial derivatives of the gravity model will only be take with respect to the spherical and J_2-J_4 terms, however, as the partials are used solely for the propagation of the covariance matrix, which is of limited accuracy due to assumptions of linearity in the state dynamics.

IMU $\Delta\mathbf{v}_s^{b_{k-1}}$ and $\bar{\mathbf{q}}_{b_{k-1}}^{b_m}$ Accumulator Function

The IMU compensation function in the absolute navigation algorithm will output $\Delta\mathbf{v}_m^{b_{m-1}}$ and $\boldsymbol{\phi}_{b_{m-1}}^{b_m}$ at a rate of either 50 Hz or 200 Hz, where the variable m denotes the number of 50 Hz or 200 Hz cycles since the last relative navigation propagation call at time $t = t_{k-1}$. For brevity, we will refer to the rate of output from the absolute navigation IMU compensation function as the high rate. The purpose of the IMU $\Delta\mathbf{v}_s^{b_{k-1}}$ and $\bar{\mathbf{q}}_{b_{k-1}}^{b_m}$ accumulator function is to take in these high rate outputs and calculate $\bar{\mathbf{q}}_{b_{k-1}}^{b_M}$ and $\Delta\mathbf{v}_s^{b_{k-1}}$ for use in the relative navigation propagation function, where $M = \frac{(t_k - t_{k-1})}{(t_m - t_{m-1})}$ is the number of high rate IMU compensator outputs between relative navigation propagation calls ($t_M = t_k$), $\bar{\mathbf{q}}_{b_{k-1}}^{b_M}$ is the left quaternion that describes the body frame rotation for the last propagation call to the current call, and $\Delta\mathbf{v}_s^{b_{k-1}}$ is the sensed change in velocity of the IMU box between propagation calls represented in the body frame as oriented during the previous relative navigation propagation call.

The vector $\boldsymbol{\phi}_{b_{m-1}}^{b_m}$ is understood to be the single axis rotation vector, with magnitude $\phi_{b_{m-1}}^{b_m} \equiv \|\boldsymbol{\phi}_{b_{m-1}}^{b_m}\|$, describing the rotation from the body frame orientation at time $t = t_{m-1}$ to the body frame orientation at time $t = t_m$. The rotation vector is zero when $m = 0$. The change in attitude from the last relative navigation propagation call to the current high rate time increment is given by

$$\bar{\mathbf{q}}_{b_{k-1}}^{b_m} = \bar{\mathbf{q}}_{b_{m-1}}^{b_m} \circledast \bar{\mathbf{q}}_{b_{k-1}}^{b_{m-1}} \tag{11}$$

where $m = 1, \ldots, M$, the initial condition is given by $\bar{\mathbf{q}}_{b_{k-1}}^{b_0} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 3} \end{bmatrix}^{\mathrm{T}}$, and

$$\bar{\mathbf{q}}_{b_{m-1}}^{b_m} = \begin{bmatrix} \cos\left(0.5\phi_{b_{m-1}}^{b_m}\right) \\ -\frac{\sin\left(0.5\phi_{b_{m-1}}^{b_m}\right)}{0.5\phi_{b_{m-1}}^{b_m}} 0.5\boldsymbol{\phi}_{b_{m-1}}^{b_m} \end{bmatrix} \tag{12}$$

The vector $\Delta\mathbf{v}_m^{b_{m-1}}$ is the sensed change in velocity of the IMU box from $t = t_{m-1}$ to $t = t_m$, represented in the body frame as oriented at time $t = t_{m-1}$. Thus, the accumulated change in sensed velocity from the last relative navigation propagation call to the current high rate time increment is given by

$$\Delta\mathbf{v}_s^{b_{k-1}} = \Delta\mathbf{v}_s^{b_{k-1}} + \left(\mathbf{T}_{b_{k-1}}^{b_{m-1}}\right)^{\mathrm{T}} \Delta\mathbf{v}_m^{b_{m-1}} \tag{13}$$

where $m = 1, \ldots, M$, the initial condition is given by $\Delta\mathbf{v}_s^{b_{k-1}} = \mathbf{0}$ at $m = 0$, and $T_{b_{k-1}}^{b_{m-1}}$ is calculated using the right hand side of Eq. (6) with $\bar{\mathbf{q}} = \bar{\mathbf{q}}_{b_{k-1}}^{b_{m-1}}$. Obviously, the $\Delta\mathbf{v}_s^{b_{k-1}}$ update must be done before the $\bar{\mathbf{q}}_{b_{k-1}}^{b_m}$ update if the need to buffer the previous version of the attitude quaternion is to be avoided.

Transformation of $\Delta\mathbf{v}_s^{b_{k-1}}$ into Inertial Frame and Reference Attitude Quaternion Update

Before updating the reference attitude quaternion, the sensed $\Delta\mathbf{v}$ since the last propagation call is thresholded and rotated into the inertial frame. If the sensed $\Delta\mathbf{v}$ is below a threshold (the value of which depends on whether the filter is in powered or coasting flight mode), then the inertial sensed $\Delta\mathbf{v}$ variable is zeroed out,

$$\Delta\mathbf{v}_s^i = \mathbf{0}_{3\times1} \tag{14}$$

and a flag is set ($I_{drag} = 1$) indicating that a drag acceleration model should be used during Orion state propagation. Otherwise, the sensed $\Delta\mathbf{v}$ is rotated into the inertial frame via

$$\Delta\mathbf{v}_s^i = \left(\mathbf{T}_b^i\right)_{k-1} \Delta\mathbf{v}_s^{b_{k-1}} = \left(\mathbf{T}_{b_{ref}}^i\right)_{k-1} \left(\mathbf{T}_b^{b_{ref}}\right)_{k-1} \Delta\mathbf{v}_s^{b_{k-1}} = \left(\mathbf{T}_{b_{ref}}^i\right)_{k-1} \Delta\mathbf{v}_s^{b_{k-1}} \tag{15}$$

where the subscript $k-1$ denotes the previous relative navigation propagation call, $\left(\mathbf{T}_{b_{ref}}^i\right)_{k-1}$ is calculated from Eq. (6), $\left(\mathbf{T}_b^{b_{ref}}\right)_{k-1}$ is calculated from Eq. (10) and is identity because $\mathbf{p}_b^{b_{ref}}$ is reset to zero before propagation, and $\Delta\mathbf{v}_s^{b_{k-1}}$ is output by the IMU accumulator function described previously. This is done at the beginning of a relative navigation filter propagation call. Next the reference attitude quaternion is updated according to

$$\left(\bar{\mathbf{q}}_{b_{ref}}^i\right)_k = \left(\bar{\mathbf{q}}_{b_{ref}}^i\right)_{k-1} \circledast \left(\bar{\mathbf{q}}_{b_{k-1}}^{b_M}\right)^* \tag{16}$$

where the subscript k denotes the current relative navigation propagation call, the superscript * denotes the quaternion complex conjugate, and $\bar{\mathbf{q}}_{b_{k-1}}^{b_M}$ is output by the IMU accumulator function described previously.

Filter State Propagation

The various components of the relative navigation filter non-linear state dynamics,

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}) = \begin{bmatrix} \dot{\mathbf{r}}_{orion}^{\mathrm{T}} & \ddot{\mathbf{r}}_{orion}^{\mathrm{T}} & (\dot{\mathbf{p}}_b^{b_{ref}})^{\mathrm{T}} & \dot{\mathbf{r}}_{targ}^{\mathrm{T}} & \ddot{\mathbf{r}}_{targ}^{\mathrm{T}} & \dot{\boldsymbol{\gamma}}^{\mathrm{T}} & \dot{\mathbf{b}}_{\alpha}^{\mathrm{T}} & \dot{b}_{\rho} & \dot{b}_{\dot{\rho}} \end{bmatrix} \tag{17}$$

are developed herein. The propagation equations are examined in four subgroups: 1) Orion position and velocity, 2) Orion attitude, 3) target position and velocity, and 4) sensor error states. Though the sensor error states were represented as continuous variables in the above equation, they are actually implemented as discrete random variables in the filter, and their difference equations will be presented in place of differential equations.

Orion Position and Velocity Orion's position and velocity are propagated in the relative navigation filter according to

$$\begin{bmatrix} \dot{\mathbf{r}}^i_{orion} \\ \ddot{\mathbf{r}}^i_{orion} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{r}}^i_{orion} \\ \mathbf{g}\left(\mathbf{r}^i_{orion}\right) + \bar{\mathbf{a}}^i_s + I_{drag} \cdot \mathbf{a}_{drag}\left(\mathbf{r}^i_{orion}\right) \end{bmatrix} \tag{18}$$

where $\mathbf{g}\left(\mathbf{r}^i_{orion}\right)$ is the acceleration due to gravity, $\bar{\mathbf{a}}^i_s \equiv \frac{\Delta \mathbf{v}^i_s}{\Delta t}$ is the average acceleration sensed by the IMU over the previous filter cycle from $t = t_{k-1}$ to $t = t_k = t_{k-1} + \Delta t$, and $\mathbf{a}_{drag}$ is a drag acceleration model. The flag I_{drag} is set to 1 if the sensed acceleration has been thresholded to zero, and is 0 otherwise.

In order to assess the effect of using the average value of the sensed acceleration in the propagation, as opposed to a 1^{st} or 2^{nd} order approximation, it is instructive to examine the Taylor series expansions of position and velocity about the time $t = t_{k-1}$:

$$\begin{aligned} \mathbf{r}_k &= \mathbf{r}_{k-1} + \dot{\mathbf{r}}_{k-1}\Delta t + \frac{1}{2}\ddot{\mathbf{r}}_{k-1}\Delta t^2 + \frac{1}{6}\left.\frac{d^3\mathbf{r}}{dt^3}\right|_{k-1}\Delta t^3 + \frac{1}{24}\left.\frac{d^4\mathbf{r}}{dt^4}\right|_{k-1}\Delta t^4 + O\left(\Delta t^5\right) \\ \dot{\mathbf{r}}_k &= \dot{\mathbf{r}}_{k-1} + \ddot{\mathbf{r}}_{k-1}\Delta t + \frac{1}{2}\left.\frac{d^3\mathbf{r}}{dt^3}\right|_{k-1}\Delta t^2 + \frac{1}{6}\left.\frac{d^4\mathbf{r}}{dt^4}\right|_{k-1}\Delta t^3 + \frac{1}{24}\left.\frac{d^5\mathbf{r}}{dt^5}\right|_{k-1}\Delta t^4 + O\left(\Delta t^5\right) \end{aligned} \tag{19}$$

where frame and vehicle notation have been suppressed for brevity. Expanding the sensed acceleration, $\mathbf{a}^i_s$, in a Taylor series about $t = t_{k-1}$ gives

$$\mathbf{a}^i_s = \mathbf{k}_0 + \mathbf{k}_1 t + \frac{1}{2}\mathbf{k}_2 t^2 + ... \tag{20}$$

where the vectors $\mathbf{k}_i$ will depend on the sensed acceleration profile at said time. Since $\bar{\mathbf{a}}^i_s = \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} \mathbf{a}^i_s dt$, the average sensed acceleration over Δt can be expressed as the true sensed acceleration plus an error, $\tilde{\mathbf{a}}$, which is given by

$$\begin{aligned} \tilde{\mathbf{a}}(t) &= \bar{\mathbf{a}}^i_s - \mathbf{a}^i_s \\ &= \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} \mathbf{a}^i_s dt - \mathbf{a}^i_s \\ &= \frac{1}{\Delta t}\left(\mathbf{k}_0\Delta t + \frac{1}{2}\mathbf{k}_1\Delta t^2 + \frac{1}{6}\mathbf{k}_2\Delta t^3 + ...\right) - \left(\mathbf{k}_0 + \mathbf{k}_1 t + \frac{1}{2}\mathbf{k}_2 t^2 + ...\right) \\ &= \mathbf{k_0}\left(\frac{\Delta t}{\Delta t} - 1\right) + \mathbf{k_1}\left(\frac{\Delta t^2}{2\Delta t} - t\right) + \mathbf{k_2}\left(\frac{\Delta t^3}{6\Delta t} - \frac{1}{2}t^2\right) + ... \\ &= \mathbf{k_1}\left(\frac{1}{2}\Delta t - t\right) + \mathbf{k_2}\left(\frac{1}{6}\Delta t^2 - \frac{1}{2}t^2\right) + ... \end{aligned} \tag{21}$$

Thus, the average sensed acceleration can be expressed as

$$\bar{\mathbf{a}}^i_s = \mathbf{a}^i_s + \tilde{\mathbf{a}} = \mathbf{a}^i_s + O\left(\Delta t\right) \tag{22}$$

Substitution of the following values,

$$\ddot{\mathbf{r}}_{k-1} \approx \mathbf{g}\left(\mathbf{r}^i_{orion}\right) + \bar{\mathbf{a}}^i_s \qquad \left.\frac{d^3\mathbf{r}}{dt^3}\right|_{k-1} \approx \dot{\mathbf{g}}_{k-1} \tag{23}$$

$$\left.\frac{d^4\mathbf{r}}{dt^4}\right|_{k-1} \approx \ddot{\mathbf{g}}_{k-1} \qquad \left.\frac{d^5\mathbf{r}}{dt^5}\right|_{k-1} \approx \left.\frac{d^3\mathbf{g}}{dt^3}\right|_{k-1} \tag{24}$$

into Eq. (19), and making use of Eq. (22) results in position integration accuracy of 2^{nd} order and velocity integration accuracy of 1^{st} order during powered flight. Since the 4^{th} order Runge-Kutta integrator that is used in the filter is equivalent to the 4^{th} order expansions given in Eq. (19), it can be seen that the assumption of constant sensed acceleration over an interval of powered flight results in a loss of accuracy of 3 orders in velocity and 2 orders in position. Propagation during non-powered flight is unaffected.

The gravitational acceleration term in Eq. (18) is the truncated gradient of the gravitational potential, which is given by

$$\begin{aligned} U &= \frac{\mu}{r} + \frac{u}{r}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{r}\right)^n P_{n,m}\left(\sin\phi\right)\left[C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right] \\ &= \frac{\mu}{r} - \frac{u}{r}\sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n P_n\left(\sin\phi\right) J_n \\ &\quad + \frac{u}{r}\sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{a_e}{r}\right)^n P_{n,m}\left(\sin\phi\right)\left[C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right] \end{aligned} \tag{25}$$

where μ is the gravitational constant of the attracting body, a_e is the mean equatorial radius, P_n is the Legendre Polynomial of degree n, $P_{n,m}$ is the Associated Legendre Function of degree n and order m, and J_n, $C_{n,m}$, and $S_{n,m}$ are spherical harmonics coefficients [2]. The coordinates used in Eq. (25) are spherical (r,ϕ,λ), and are calculated in terms of the position vector represented in the Earth-Fixed Earth Centered (ECEF) frame, which is calculated by

$$\mathbf{r}_{orion}^{ecef} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = T_i^{ecef}\mathbf{r}_{orion}^{i} \tag{26}$$

where T_i^{ecef} is the RNP matrix. From there, it is easy to calculate the radius, geocentric latitude and longitude, which are given by

$$r = \|\mathbf{r}_{orion}^{ecef}\| \qquad \phi = \sin^{-1}\frac{\xi_3}{r} \qquad \lambda = \tan^{-1}\frac{\xi_2}{\xi_1} \tag{27}$$

respectively. The Legendre Polynomials are given by

$$\begin{aligned} P_0\left(\sin\phi\right) &= 1 \\ P_1\left(\sin\phi\right) &= \sin\phi \\ P_n\left(\sin\phi\right) &= \left[\left(2n-1\right)\sin\phi P_{n-1}\left(\sin\phi\right) - \left(n-1\right)P_{n-2}\left(\sin\phi\right)\right]/n \end{aligned} \tag{28}$$

and the m^{th}-order partial derivative of P_n with respect to $\sin\phi$, denoted $P_n^m = \frac{\partial^m P_n}{\partial(\sin\phi)^m}$, is

$$\begin{aligned} P_0^1 &= 0 \\ P_1^1 &= 1 \\ P_n^m &= P_{n-2}^m + \left(2n-1\right)P_{n-1}^{m-1} \end{aligned} \tag{29}$$

where $m \geq 1$ [4]. Following the formulation presented in [4], the Associated Legendre Functions are given by

$$P_{n,m} = \left(1-\sin^2\phi\right)^{m/2} P_n^m = \left(\frac{r^2-\xi_3^2}{r^2}\right)^{m/2} P_n^m = \frac{\rho^m}{r^m}P_n^m \tag{30}$$

where $\rho = \sqrt{r^2 - \xi_3^2}$. Let

$$\begin{aligned} C_m &\equiv \rho^m \cos m\lambda \\ S_m &\equiv \rho^m \sin m\lambda \\ B_{n,m} &\equiv C_{n,m} C_m + S_{n,m} S_m \end{aligned} \tag{31}$$

Then, Eq. (25) can be rewritten as

$$U = \frac{\mu}{r} + \frac{u}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n \frac{P_n^m (\sin\phi)}{r^m} B_{n,m} \tag{32}$$

which is the result arrived at by Gottlieb [4]. Note that the zonal harmonic terms correspond to $m = 0$, so that $J_n = -B_{n,0}$.

The gradient of Eq. (32) with respect to $\mathbf{r}_{orion}^{ecef}$ is given as

$$\nabla U = \frac{\partial U}{\partial r} \frac{\partial r}{\partial \mathbf{r}_{orion}^{ecef}} + \frac{\partial U}{\partial (\sin\phi)} \frac{\partial (\sin\phi)}{\partial \mathbf{r}_{orion}^{ecef}} + \frac{\partial U}{\partial B_{n,m}} \frac{\partial B_{n,m}}{\partial \mathbf{r}_{orion}^{ecef}} \tag{33}$$

where

$$\begin{aligned} \frac{\partial U}{\partial r} &= -\frac{\mu}{r^2} - \frac{\mu}{r^2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n \frac{(n+m+1)}{r^m} P_n^m (\sin\phi) B_{n,m} \\ \frac{\partial U}{\partial (\sin\phi)} &= \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n \frac{P_n^{m+1} (\sin\phi)}{r^m} B_{n,m} \\ \frac{\partial U}{\partial B_{n,m}} &= \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n \frac{P_n^m (\sin\phi)}{r^m} \end{aligned} \tag{34}$$

and

$$\begin{aligned} \frac{\partial r}{\partial \mathbf{r}_{orion}^{ecef}} &= \frac{1}{r} \mathbf{r}_{orion}^{ecef} \\ \frac{\partial \sin\phi}{\partial \mathbf{r}_{orion}^{ecef}} &= \frac{1}{r} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\xi_3}{r^3} \mathbf{r}_{orion}^{ecef} \\ \frac{\partial B_{n,m}}{\partial \mathbf{r}_{orion}^{ecef}} &= \frac{m B_{n,m}}{\rho} \frac{\partial \rho}{\partial \mathbf{r}_{orion}^{ecef}} - m \left(C_{n,m} S_m - S_{n,m} C_m\right) \frac{\partial \lambda}{\partial \mathbf{r}_{orion}^{ecef}} \\ &= m \begin{bmatrix} \frac{\xi_1}{\rho^2} \left(C_{n,m} C_m + S_{n,m} S_m\right) + \frac{\xi_2}{\rho^2} \left(C_{n,m} S_m - S_{n,m} C_m\right) \\ \frac{\xi_2}{\rho^2} \left(C_{n,m} C_m + S_{n,m} S_m\right) - \frac{\xi_1}{\rho^2} \left(C_{n,m} S_m - S_{n,m} C_m\right) \\ 0 \end{bmatrix} \end{aligned} \tag{35}$$

The term $\frac{\partial B_{n,m}}{\partial \mathbf{r}_{orion}^{ecef}}$ can be simplified considerably by noting that [4]

$$C_m = C_1 C_{m-1} - S_1 S_{m-1} \qquad S_m = S_1 C_{m-1} + C_1 S_{m-1} \tag{36}$$

and that

$$C_1 = \rho \cos\lambda = \rho \cdot \frac{\xi_1}{\rho} = \xi_1 \qquad S_1 = \rho \sin\lambda = \rho \cdot \frac{\xi_2}{\rho} = \xi_2$$

$$\left(C_1^2+S_1^2\right)=\rho^2 \tag{37}$$

Then, the aforementioned partial derivative becomes

$$\begin{aligned}
\frac{\partial B_{n,m}}{\partial \mathbf{r}_{orion}^{ecef}} &= m\begin{bmatrix} \frac{1}{\rho^2}\left[C_{n,m}\left(C_1^2 C_{m-1}-C_1 S_1 S_{m-1}+S_1^2 C_{m-1}+S_1 C_1 S_{m-1}\right)\right. \\ \left. +S_{n,m}\left(C_1 S_1 C_{m-1}+C_1^2 S_{m-1}-C_1 S_1 C_{m-1}+S_1^2 S_{m-1}\right)\right] \\ \\ \frac{1}{\rho^2}\left[C_{n,m}\left(S_1 C_1 C_{m-1}-S_1^2 S_{m-1}-C_1 S_1 C_{m-1}-C_1^2 S_{m-1}\right)\right. \\ \left. +S_{n,m}\left(S_1^2 C_{m-1}+C_1 S_1 S_{m-1}+C_1^2 C_{m-1}-C_1 S_1 S_{m-1}\right)\right] \\ \\ 0 \end{bmatrix} \\
&= m\begin{bmatrix} \left(C_{n,m}C_{m-1}+S_{n,m}S_{m-1}\right) \\ -\left(C_{n,m}S_{m-1}-S_{n,m}C_{m-1}\right) \\ 0 \end{bmatrix}
\end{aligned} \tag{38}$$

Finally, the gravitational acceleration term in Eq. (18) can be computed by pre-multiplying Eq. (33) by the transpose of the RNP matrix:

$$\mathbf{g}\left(\mathbf{r}_{orion}^{i}\right)=\mathbf{T}_{ecef}^{i}\nabla U\left(\mathbf{r}_{orion}^{ecef}\right) \tag{39}$$

Orion Attitude MRP The derivative of the Modified Rodriguez Parameter (MRP) that is given in Eq. (7) is

$$\dot{\mathbf{p}}_b^{b_{ref}}=\frac{4\delta\dot{\mathbf{q}}}{1+\delta q_0}-\frac{4\delta\mathbf{q}}{\left(1+\delta q_0\right)^2}\delta\dot{q}_0 \tag{40}$$

The deviation quaternion, $\delta\bar{\mathbf{q}}\equiv\delta\bar{\mathbf{q}}_b^{b_{ref}}$, is defined to be

$$\delta\bar{\mathbf{q}}=\left(\bar{\mathbf{q}}_{b_{ref}}^{i}\right)^*\circledast\bar{\mathbf{q}}_b^i \tag{41}$$

so that

$$\delta\dot{\bar{\mathbf{q}}} = \left(\dot{\bar{\mathbf{q}}}_{b_{ref}}^{i}\right)^*\circledast\bar{\mathbf{q}}_b^i+\left(\bar{\mathbf{q}}_{b_{ref}}^{i}\right)^*\circledast\dot{\bar{\mathbf{q}}}_b^i=\begin{bmatrix} 0 \\ \delta\mathbf{q}\times\boldsymbol{\omega}^b \end{bmatrix} \tag{42}$$

where it has been assumed that the actual and reference quaternions are propagated with the same angular velocity vector as represented in the body frame, the gyro error is entered in the process noise. Substituting the results of Eq. (42) into Eq. (40) yields

$$\dot{\mathbf{p}}_b^{b_{ref}}=\frac{4\delta\mathbf{q}\times\boldsymbol{\omega}^b}{1+\delta q_0}=\mathbf{p}_b^{b_{ref}}\times\boldsymbol{\omega}^b \tag{43}$$

The propagation of the MRP in the filter requires the evaluation of Eq. (43) along the nominal trajectory. After each measurement update, the reference quaternion is updated using the MRP, and the MRP is reset to its nominal value of zero. Therefore, the propagation equation used in the filter is

$$\dot{\mathbf{p}}_b^{b_{ref}}=\left[\mathbf{p}_b^{b_{ref}}\times\boldsymbol{\omega}^b\right]_{\mathbf{p}_b^{b_{ref}}=\mathbf{0}}=\mathbf{0} \tag{44}$$

That is to say, the MRP is not propagated by the filter.

Target Position and Velocity The propagation of the target position and velocity depend solely on gravitational acceleration and drag,

$$\begin{bmatrix} \dot{\mathbf{r}}^i_{targ} \\ \ddot{\mathbf{r}}^i_{targ} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{r}}^i_{targ} \\ \mathbf{g}\left(\mathbf{r}^i_{targ}\right) + \mathbf{a}_{drag}\left(\mathbf{r}^i_{targ}\right) \end{bmatrix} \tag{45}$$

where $\mathbf{g}\left(\mathbf{r}^i_{targ}\right)$ is given by substituting $\mathbf{r}^i = \mathbf{r}^i_{targ}$ into Eq. (39).

Sensor Misalignment and Bias States The states that model sensor misalignment and biases are modeled as independent exponentially correlated random variables (ECRV's), which are 1^{st} order gauss-markov processes. Discrete ECRV's are used in the filter for ease of propagation and state transition matrix computation. The update equation for a discrete ECRV from time $t = t_{k-1}$ to time $t = t_k$, with correlation time constant $\tau = 1/\beta$, is

$$b_k = e^{-\beta(t_k - t_{k-1})} b_{k-1} + w_{k-1} \tag{46}$$

where $\mathrm{E}\left\{w_{k-1}\right\} = 0$ is used to update the value of the state,

$$\mathrm{E}\left\{w^2_{k-1}\right\} = \sigma^2\left[1 - e^{-2\beta(t_k - t_{k-1})}\right] \tag{47}$$

and b is any of the zero mean sensor error states ($b_{\gamma_1}, b_{\gamma_2}, b_{\gamma_3}, b_{\alpha_h}, b_{\alpha_v}, b_\rho, b_{\dot\rho}$), with $\sigma^2 = E\left[b^2\right]$, which is constant.

State Transition Matrix Propagation and Covariance Matrix Update

The filter covariance matrix, $\mathbf{P}$, is updated from one time step to the next according to

$$\mathbf{P}(t_k) = \mathbf{\Phi}\left(t_k, t_{k-1}\right)\mathbf{P}(t_{k-1})\mathbf{\Phi}^{\mathrm{T}}\left(t_k, t_{k-1}\right) + \mathbf{Q} \tag{48}$$

where $\mathbf{\Phi}\left(t_k, t_{k-1}\right)$ is the state transition matrix and $\mathbf{Q}$ is a covariance matrix (assumed constant over the time step) that is the result of process noise. Linearized state dynamics are assumed in the definition of the state transition matrix.

The differential equation describing the evolution of the state transition matrix is

$$\dot{\mathbf{\Phi}}\left(t, t_{k-1}\right) = \mathbf{F}\left(t\right)\mathbf{\Phi}\left(t, t_{k-1}\right) \tag{49}$$

where

$$\mathbf{F} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}^*} \tag{50}$$

and $\mathbf{\Phi}\left(t_{k-1}, t_{k-1}\right) = \mathbf{I}_{n\times n}$. The nominal state, along which the partials are evaluated, is denoted by $\mathbf{X}^*$. The matrix of partials, $\mathbf{F}$, is found from Eqs. (17) and (50) to be

$$\mathbf{F} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times7} \\ \frac{\partial \ddot{\mathbf{r}}^i_{orion}}{\partial \mathbf{r}^i_{orion}} & \mathbf{O}_{3\times3} & \frac{\partial \ddot{\mathbf{r}}^i_{orion}}{\partial \mathbf{p}_b^{b_{ref}}} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times7} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \frac{\partial \dot{\mathbf{p}}_b^{b_{ref}}}{\partial \mathbf{p}_b^{b_{ref}}} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times7} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{O}_{3\times7} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \frac{\partial \ddot{\mathbf{r}}^i_{targ}}{\partial \mathbf{r}^i_{targ}} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times7} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} & -\operatorname{diag}(\boldsymbol{\beta}) \end{bmatrix}_{\mathbf{X}=\mathbf{X}^*} \tag{51}$$

where $\boldsymbol{\beta} = \begin{bmatrix} \beta_\gamma & \beta_\gamma & \beta_\gamma & \beta_\alpha & \beta_\alpha & \beta_\rho & \beta_{\dot{\rho}} \end{bmatrix}^{\mathrm{T}}$ and β_γ, β_α, β_ρ, and $\beta_{\dot{\rho}}$ are the inverses of the times constants on sensor misalignment, angle bias, range bias, and range rate bias states, respectively. All partial derivatives are calculated along the nominal trajectory.

The state transition matrix is calculated by dividing the matrix up into sub-matrices to take advantage of its sparse matrix properties. By comparing Eqs (49) and (51), recalling that $\mathbf{\Phi}(t_{k-1}, t_{k-1}) = \mathbf{I}_{n\times n}$, and using the notation

$$\mathbf{\Phi}_{i-j,m-n} = \begin{bmatrix} \phi_{im} & \cdots & \phi_{in} \\ \vdots & \ddots & \vdots \\ \phi_{jm} & \cdots & \phi_{jn} \end{bmatrix} \tag{52}$$

where $\mathbf{\Phi}(t, t_{k-1}) = \{\phi_{ij}\}$, it becomes evident that only the submatrices $\mathbf{\Phi}_{1-9,1-9}$, $\mathbf{\Phi}_{10-15,10-15}$, and $\mathbf{\Phi}_{16-22,16-22}$ are non-zero, with the last being a diagonal matrix.

MEASUREMENT PROCESSING

The filter processes the measurements of four sensors during RPOD in LEO and three sensors during RPOD in LLO, where GPS is unavailable. A state interpolation algorithm has been developed to aid in the processing of latent measurements. In this section, the procedure for processing latent measurements is addressed first, followed by a discussion of inertial measurement processing, and then relative measurement processing. The IMU measurements are not included in this section because the gyro and accelerometer outputs are used for dead-reckoning, and so are not processed as measurements in the traditional filter sense.

To account for the fact that the state can vary in size, and in an effort to keep the notation succinct, the measurement partial derivative matrices in what follows will be presented as $m \times n$ matrices, where m is the number of measurements and n is the number of active states (out of a maximum of 22). All measurements are processed as scalars with independent noise characteristics unless otherwise noted (i.e. the two components of a bearing measurement are processed separately as scalars). Also, all measurement model and partial derivative calculations are performed using the nominal state, which is only updated after all measurements have been processed. This means that $\mathbf{p}_b^{b_{ref}} = \mathbf{0}_{3\times1}$ and $\mathbf{T}_{b_{ref}}^b = \mathbf{I}_{3\times3}$ when the terms appear in equations below that are evaluated along the nominal trajectory.

Procedure for Processing Latent Measurements

A sensor will require some finite amount of time to process a measurement before sending the finished product to the relative navigation algorithm for filtering. Thus, a situation will arise where the filter has propagated its state and covariance to time $t = t_k$ from time $t = t_{k-1}$, and is subsequently given a measurement to be filtered (denoted by subscript j) that corresponds to time $t = t_j$, where

$$t_{k-1} < t_j \leq t_k \tag{53}$$

If $\Delta t = t_j - t_k$ is not insignificant, the time difference between the measurement and the filter state and covariance will need to be accounted for during filtering in order to accurately process the measurement. This can be done in much the same way a batch filter operates (see pages 196-197 of [2] where scalar measurements are used). If the measurement at time $t = t_j$ is denoted as Y_j, the nominal filter state at that time is given by $\mathbf{X}_j^*$ ($*$ denotes the nominal), and the filter measurement model is denoted as $h_j(\mathbf{X}, t_j)$, then the scalar residual of the measurement is given by

$$y_j - \mathbf{H}_j \hat{\mathbf{x}}_k(-) = Y_j - h_j\left(\mathbf{X}_j^*, t_j\right) - \tilde{\mathbf{H}}_j \mathbf{\Phi}(t_j, t_k) \hat{\mathbf{x}}_k(-) \tag{54}$$

where $\hat{\cdot}$ denotes an estimated value,

$$\begin{aligned} y_j &= Y_j - h_j\left(\mathbf{X}_j^*, t_j\right) \\ \hat{\mathbf{x}}_k(-) &= \hat{\mathbf{X}}_j(-) - \mathbf{X}_j^* \end{aligned} \tag{55}$$

and

$$\tilde{\mathbf{H}}_j = \left.\frac{\partial h_j(\mathbf{X}, t_j)}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}_j^*} \tag{56}$$

is the partial derivative matrix mapping the measurement to the state at time $t = t_j$. The measurement partials that are used in the update, which map the measurement to the state at time $t = t_k$, are given by

$$\mathbf{H}_j = \tilde{\mathbf{H}}_j \mathbf{\Phi}(t_j, t_k) \tag{57}$$

Eq. (57) was derived by noting that

$$\tilde{\mathbf{H}}_j \hat{\mathbf{x}}_j = \tilde{\mathbf{H}}_j \mathbf{\Phi}(t_j, t_k) \hat{\mathbf{x}}_k = \mathbf{H}_j \hat{\mathbf{x}}_k \tag{58}$$

Assuming the measurement residual and partial derivative matrix can be calculated, the filter vector measurement update then proceeds according to

$$\begin{aligned} \mathbf{K}_j &= \mathbf{P}_k(-) \mathbf{H}_j^{\mathrm{T}} \left(\mathbf{H}_j \mathbf{P}_k(-) \mathbf{H}_j^{\mathrm{T}} + \mathbf{R}_j\right)^{-1} \\ \hat{\mathbf{X}}_k(+) &= \hat{\mathbf{X}}_k(-) + \mathbf{K}_j \left[\mathbf{Y}_j - \mathbf{h}_j\left(\hat{\mathbf{X}}_k(-)\right)\right] \\ \mathbf{P}_k(+) &= \left(\mathbf{I}_{n\times n} - \mathbf{K}_j \mathbf{H}_j\right) \mathbf{P}_k(-) \end{aligned} \tag{59}$$

Therefore, the unknown quantities in Eqs. (54)-(57) that are needed to update the state at time $t = t_k$ with a measurement from time $t = t_j$ are the nominal state at the measurement time, $\mathbf{X}_j^*$, and the state transition matrix relating the two times, $\mathbf{\Phi}(t_j, t_k)$. Given those values, $h_j\left(\mathbf{X}_j^*, t_j\right)$ and $\tilde{\mathbf{H}}_j$ can be calculated.

The state transition matrix can be approximated as:

$$\mathbf{\Phi}(t_j, t_k) \approx \mathbf{I}_{n\times n} + \mathbf{F}_k \Delta t \tag{60}$$

where, from Eq. (50)

$$\mathbf{F}_k = \left.\frac{\partial \mathbf{f}(\mathbf{X}, t_k)}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}_k^*} \tag{61}$$

and $\Delta t = t_j - t_k$.

The nominal state at time $t = t_j$ can be approximated by truncating its Taylor series expansion about $t = t_k$ after 2^{nd}-order:

$$\begin{aligned}
\mathbf{X}_j^* &\approx \mathbf{X}_k^* + \dot{\mathbf{X}}_k^* \Delta t + \frac{1}{2}\ddot{\mathbf{X}}_k^* (\Delta t)^2 \\
&= \mathbf{X}_k^* + \mathbf{f}(\mathbf{X}_k^*, t_k)\Delta t + \frac{1}{2}\left(\left.\frac{\partial \mathbf{f}(\mathbf{X}, t_k)}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}_k^*} \dot{\mathbf{X}}_k^* + \left.\frac{\partial \mathbf{f}(\mathbf{X}_k^*, t)}{\partial t}\right|_{t=t_k}\right)(\Delta t)^2 \\
&= \mathbf{X}_k^* + \mathbf{f}_k \Delta t + \frac{1}{2}\mathbf{F}_k \mathbf{f}_k (\Delta t)^2
\end{aligned} \tag{62}$$

The 2^{nd}-order state approximation and 1^{st}-order state transition matrix approximation only require one evaluation of the non-linear dynamics and one evaluation of the dynamics partials. The same value of $\mathbf{f}_k$ and $\mathbf{F}_k$ can be used with different Δt's to process all the measurements available for processing at filter time step k. This is in contrast to a 4^{th}-order Runge-Kutta backward propagation of the state and covariance matrix, which would require at least four evaluations of each quantity for each measurement to be processed (assuming the measurements occur at different times). So long as measurement latencies are small or occur during coasting flight, the extrapolation scheme proposed above should be accurate enough, and require much less processing time than the alternative.

Note that some measurements will require the value of $\bar{\mathbf{q}}_{b_{ref}}^i$, which is not included in the state, at the time of the measurement. Since the filter models the angular rate over the last filter interval as a constant which is equal to the average rate, the value of $\bar{\mathbf{q}}_{b_{ref}}^i$ at time $t = t_j$ can be approximated by

$$\left(\bar{\mathbf{q}}_{b_{ref}}^i\right)_j = \left(\bar{\mathbf{q}}_{b_{ref}}^i\right)_k \circledast \bar{\mathbf{q}}_{b_j}^{b_k} \tag{63}$$

where

$$\bar{\mathbf{q}}_{b_j}^{b_k} = \begin{bmatrix} \cos\left(0.5\phi_{b_j}^{b_k}\right) \\ -\sin\left(0.5\phi_{b_j}^{b_k}\right)\boldsymbol{\phi}_{b_j}^{b_k}/\phi_{b_j}^{b_k} \end{bmatrix} \tag{64}$$

and $\boldsymbol{\phi}_{b_j}^{b_k} = \bar{\boldsymbol{\omega}}^b \cdot (t_k - t_j)$.

Inertial Measurements

The GPS and star tracker attitude measurements are processed to keep the inertial filter states from wandering off, which could result in divergence in the relative state due to a bad estimate of gravity or improper incorporation of relative bearing measurements.

GPS Inertial Position and Velocity Measurements It is anticipated that the absolute navigation system on-board Orion will convert pseudo range and delta range measurements from the selected GPS unit to a deterministic inertial position and velocity estimate for the relative navigation algorithm to use. Since the relative navigation filter does not require an extremely accurate absolute state estimate, it was decided that the occasional processing of the aforementioned deterministic measurements would be sufficient, and the more complex processing of raw GPS measurements could be avoided in the relative filter. Also, in the effort to avoid modeling the time varying bias inherent in the GPS measurements, said measurements will only be incorporated over intervals lengthy enough in time that consecutive measurements will be essentially uncorrelated. GPS measurements are not processed during final approach to prevent filter jitter.

The analytic model of the GPS measurements used by the filter is simply

$$\mathbf{h}_{gps}\left(t,\mathbf{X}\right)=\begin{bmatrix}\mathbf{r}^{i}_{orion}\\ \\ \dot{\mathbf{r}}^{i}_{orion}\end{bmatrix} \tag{65}$$

The partial derivative matrix corresponding to the measurement model in Eq. (65) is easily derived as

$$\tilde{\mathbf{H}}_{gps} = \left.\frac{\partial\mathbf{h}_{gps}}{\partial\mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}^*} = \begin{bmatrix}\mathbf{I}_{6\times6} & \mathbf{O}_{6\times(n-6)}\end{bmatrix} \tag{66}$$

Star Tracker Inertial Attitude Measurements

The selected star tracker provides a measurement of its case-to-inertial attitude quaternion, $\left(\bar{\mathbf{q}}^{i}_{c}\right)_{meas}$ (in the form of a left quaternion). Due to the fact that the 4 elements of a quaternion are not independent, the incoming quaternion is converted to a MRP prior to processing by the filter

$$\left(\mathbf{p}^{b_{ref}}_{b}\right)_{m}=\frac{4\mathbf{q}_{m}}{1+q_{0_m}} \tag{67}$$

where

$$\bar{\mathbf{q}}_{m}=\left(\bar{\mathbf{q}}^{i}_{b_{ref}}\right)^{*}\circledast\left(\bar{\mathbf{q}}^{i}_{c}\right)_{meas}\circledast\bar{\mathbf{q}}^{c_{ref}}_{b} \tag{68}$$

and $\bar{\mathbf{q}}^{c_{ref}}_{b}$ is the specified transformation from the body frame to the sensor case frame (note that there is a sensor misalignment which is not yet modeled here; it is anticipated that the misalignment dependency will be added).

Though the noise characteristics of the three elements of the measured MRP are not independent, the filter processes them as scalar independent measurements. The consequence of this approach is that the filter tends to be conservative because the correlations between the MRP components, which contain useful information about the measurement, are ignored.

The analytic model of the attitude MRP measurement used by the filter is given by

$$\mathbf{h}_{st_{att}}\left(t,\mathbf{X}\right)=\mathbf{p}^{b_{ref}}_{b} \tag{69}$$

which is nominally zero. The resulting partial derivative matrix is simply

$$\tilde{\mathbf{H}}_{st_{att}} = \left.\frac{\partial \mathbf{h}_{st_{att}}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}^*} = \left[\begin{array}{ccc} \mathbf{O}_{3\times 6} & \mathbf{I}_{3\times 3} & \mathbf{O}_{3\times(n-9)} \end{array}\right] \tag{70}$$

Relative Measurements

Bearing measurements from the star tracker, communications system, and Vision Navigation Sensor (VNS) are processed to improve the relative state accuracy. Range measurements are available from the VNS and the communications system. The communications system also provides range rate measurements. All measurements of the same type are processed in the same manner, so measurement types (bearing, range, and range rate) are addressed below, rather than sensor types.

Bearing Measurements The star tracker, communications system, and VNS all provide bearing measurements at different ranges. Although the possibility of overlap exist between sensor ranges, the filter will only process bearing updates from the sensor selected by the relative navigation executive. The bearing measurements correspond to a horizontal and vertical bearing in the sensor frame. In the case of the star tracker and VNS, this bearing formulation corresponds to the bearing along to two dimensions of the image plane, and so the measurement noise can be assumed to be independent. It is not yet known if the same assumption will hold for the case of the bearing measurements generated by the communications system.

Assuming the z-axis of the sensor case frame is along the sensor boresight, the analytic model for the two bearing measurements used by the filter is

$$\mathbf{h}_{bearing}(t, \mathbf{X}) = \boldsymbol{\alpha} + \mathbf{b}_{\alpha} = \left[\begin{array}{c} \alpha_h \\ \alpha_v \end{array}\right] + \left[\begin{array}{c} b_{\alpha_h} \\ b_{\alpha_v} \end{array}\right] = \left[\begin{array}{c} \tan^{-1}\left(\frac{x}{z}\right) + b_{\alpha_h} \\ \tan^{-1}\left(\frac{y}{z}\right) + b_{\alpha_v} \end{array}\right] \tag{71}$$

where α_v is considered the vertical component of the bearing, α_h is the horizontal component, and x, y, and z are the components of the position of the target with respect to the sensor, represented in the sensor case frame:

$$\begin{aligned} \mathbf{r}^c_{tf/c} &= \left[\begin{array}{c} x \\ y \\ z \end{array}\right] \\ &= \mathbf{T}^c_{c_{ref}} \mathbf{T}^{c_{ref}}_s \left[\mathbf{T}^s_b \mathbf{T}^b_{b_{ref}} \mathbf{T}^{b_{ref}}_i \left(\mathbf{r}^i_{targ} + \mathbf{r}^i_{tf/targ} - \mathbf{r}^i_{orion}\right) - \left(\mathbf{r}^s_{c/s} - \mathbf{r}^s_{orion/s}\right)\right] \end{aligned} \tag{72}$$

In Eq. (72), $\mathbf{T}^{c_{ref}}_s$ is the transformation matrix defining the orientation of the reference sensor frame with respect the the Orion structural frame, $\mathbf{T}^c_{c_{ref}}$ is the transformation matrix containing the estimate of the sensor frame misalignment from the reference sensor frame, $\mathbf{r}^s_{c/s}$ is the location of the sensor case with respect to the Orion structural frame, $\mathbf{r}^s_{orion/s}$ is the location of the navigation reference point of the Orion vehicle with respect to the Orion structural frame (recall that the location of the selected IMU is the navigation reference point), and $\mathbf{r}^i_{tf/targ}$ is the location of the feature on the target vehicle that the sensor is tracking, given with respect to the target vehicle c.g. and expressed in the inertial frame. The value of $\mathbf{r}^i_{tf/targ}$ will only be non-zero for bearing measurements from the VNS, due to the proximity of the vehicle to the target when the VNS is operational. A subfunction will calculate $\mathbf{r}^i_{tf/targ}$ based on the assumed target vehicle attitude and known target feature geometry.

The partial derivative matrix corresponding to the bearing model of Eq. (71) is

$$\begin{aligned}\tilde{\mathbf{H}}_{bearing} &= \left.\frac{\partial \mathbf{h}_{bearing}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}^*} \\ &= \left[\begin{array}{cccccccc} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{r}^c_{tf/c}} \left(\frac{\partial \mathbf{r}^c_{tf/c}}{\partial \mathbf{r}^i_{orion}}\right. & \mathbf{O}_{2\times 3} & \frac{\partial \mathbf{r}^c_{tf/c}}{\partial \mathbf{p}_b^{b_{ref}}} & \frac{\partial \mathbf{r}^c_{tf/c}}{\partial \mathbf{r}^i_{targ}} & \mathbf{O}_{2\times 3} & \left.\frac{\partial \mathbf{r}^c_{tf/c}}{\partial \boldsymbol{\gamma}}\right) & I_{2\times 2} & \mathbf{O}_{2\times(n-20)} \end{array}\right]_{\mathbf{X}=\mathbf{X}^*} \end{aligned} \quad (73)$$

under the assumptions that, nominally, $\mathbf{p}_b^{b_{ref}} = \mathbf{0}_{3\times 1}$ and the sensor misalignment angles, $\boldsymbol{\gamma}$, are small so that the transformation matrix $\mathbf{T}^c_{c_{ref}}$ can be approximated as

$$\mathbf{T}^c_{c_{ref}} \approx \mathbf{I}_{3\times 3} - [\boldsymbol{\gamma}\times] \quad (74)$$

Range Measurements The communication system and the VNS will both provide a range measurement at different ranges to the target vehicle. Again, the relative navigation algorithm will only process range from one sensor at a time, and the selection occurs at the relative navigation executive level. The range measurement is independent of frame in which the relative position vector is represented, and is calculated as the distance from the sensor to a known feature on the target vehicle. The analytic model for the range measurement used by the filter is

$$h_\rho(t, \mathbf{X}) = \rho + b_\rho = \sqrt{\left(\mathbf{r}^i_{tf/c}\right)^{\mathrm{T}} \mathbf{r}^i_{tf/c}} + b_\rho \quad (75)$$

where

$$\mathbf{r}^i_{tf/c} = \mathbf{r}^i_{targ} + \mathbf{r}^i_{tf/targ} - \mathbf{r}^i_{orion} - \mathbf{T}^i_{b_{ref}} \mathbf{T}_b^{b_{ref}} \mathbf{T}^b_s \left(\mathbf{r}^s_{c/s} - \mathbf{r}^s_{orion/s}\right) \quad (76)$$

The various terms in Eq. (76) that are not derived from elements of the state vector are defined in Section . The partial derivative matrix corresponding to the range model of Eq. (75) is given by

$$\begin{aligned}\tilde{\mathbf{H}}_\rho &= \left.\frac{\partial h_\rho}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}^*} \\ &= \left[\begin{array}{cccccccc} \frac{\partial \rho}{\partial \mathbf{r}^i_{tf/c}} \left(-\mathbf{I}_{1\times 3}\right. & \mathbf{O}_{1\times 3} & \frac{\partial \mathbf{r}^i_{tf/c}}{\partial \mathbf{p}_b^{b_{ref}}} & \left.\mathbf{I}_{1\times 3}\right) & \mathbf{O}_{1\times 8} & 1 & \mathbf{O}_{1\times(n-21)} \end{array}\right]_{\mathbf{X}=\mathbf{X}^*} \end{aligned} \quad (77)$$

under the assumption that, nominally, $\mathbf{p}_b^{b_{ref}} = \mathbf{0}_{3\times 1}$.

Range Rate Measurements The communication system will provide a range rate measurement within a specified range from the target vehicle. The relative navigation algorithm will only process the range rate measurement when also processing the range measurement from the communications system. The selection occurs at the relative navigation executive level. It is easiest, given the nature of the filter state, to compute range rate using vectors expressed in the inertial frame. Thus, the analytic model for the range rate measurement used by the filter is

$$h_{\dot\rho}(t, \mathbf{X}) = \dot\rho + b_{\dot\rho} = \frac{1}{\rho}\left(\mathbf{r}^i_{tf/c}\right)^{\mathrm{T}} \dot{\mathbf{r}}^i_{tf/c} + b_{\dot\rho} \quad (78)$$

where, borrowing from Eqs. (10) and (43), and assuming $\mathbf{p}_b^{b_{ref}} = \mathbf{0}_{3\times 1}$ nominally,

$$\begin{aligned}\dot{\mathbf{r}}^i_{tf/c} &= \dot{\mathbf{r}}^i_{targ} - \dot{\mathbf{r}}^i_{orion} - \mathbf{T}^i_{b_{ref}}\left[\boldsymbol{\omega}^b\times\right]\mathbf{r}^{b_{ref}}_{c/orion} - \mathbf{T}^i_{b_{ref}}\left[\left[\left(\mathbf{p}_b^{b_{ref}}\times\boldsymbol{\omega}^b\right)\times\right] + O\left(\left(\mathbf{p}_b^{b_{ref}}\right)^2\right)\right]\mathbf{r}^b_{c/orion} \\ &= \dot{\mathbf{r}}^i_{targ} - \dot{\mathbf{r}}^i_{orion} - \mathbf{T}^i_{b_{ref}}\left[\boldsymbol{\omega}^b\times\right]\mathbf{r}^{b_{ref}}_{c/orion} \end{aligned} \quad (79)$$

The differential equation describing the rate of change of a transformation matrix is taken from [3]. The partial derivative matrix corresponding to the range rate model of Eq. (78) is given by

$$\begin{aligned}\tilde{\mathbf{H}}_{\dot\rho} &= \left.\frac{\partial h_{\dot\rho}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X}^*} \\ &= \left[\begin{array}{ccccccc} \frac{\partial \dot\rho}{\partial \mathbf{r}^i_{orion}} & \frac{\partial \dot\rho}{\partial \dot{\mathbf{r}}^i_{orion}} & \frac{\partial \dot\rho}{\partial \mathbf{p}_b^{b_{ref}}} & \frac{\partial \dot\rho}{\partial \mathbf{r}^i_{targ}} & \frac{\partial \dot\rho}{\partial \dot{\mathbf{r}}^i_{targ}} & \mathbf{O}_{1\times 6} & 1 \end{array}\right]_{\mathbf{X}=\mathbf{X}^*}\end{aligned} \tag{80}$$

NUMERICAL RESULTS

This section contains the preliminary numerical results obtained with the NASA ANTARES simulation. The case studied is RPOD with the Inernational Space Station. Figures 1–2 illustrate the performance of the filter, which is derived from 500 Monte Carlo simulations. The results are shown in the LVLH frame, where the x axis is downrange, y is the out-of-plane, and z is radial. It can be seen that the downrange grows rapidly, this is fact is anticipated to be mitigating by using a more sophisticated drag model. Once the communication is established (within 30 kilometers), the errors are reduced.

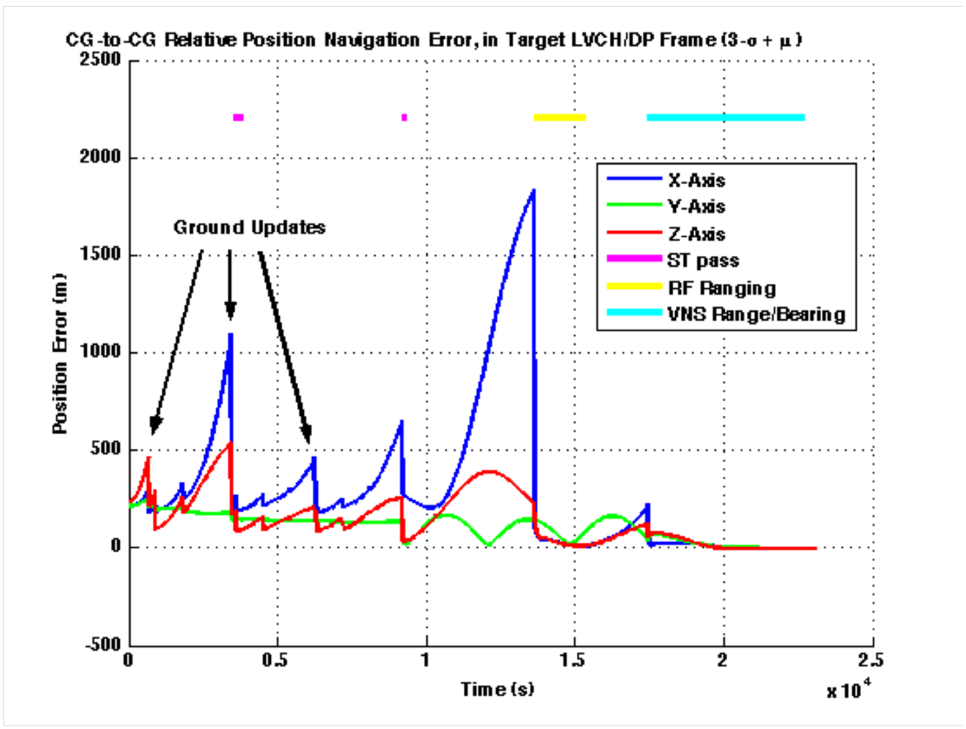

Figure 1 Relative position errors

CONCLUSIONS

The preliminary design for the Orion relative navigation filter was presented. A dual state formulation was selected, with inertial states of both Orion and the target. It was shown thorough the use of a high fidelity simulation that the design is adequate to support RPOD operations.

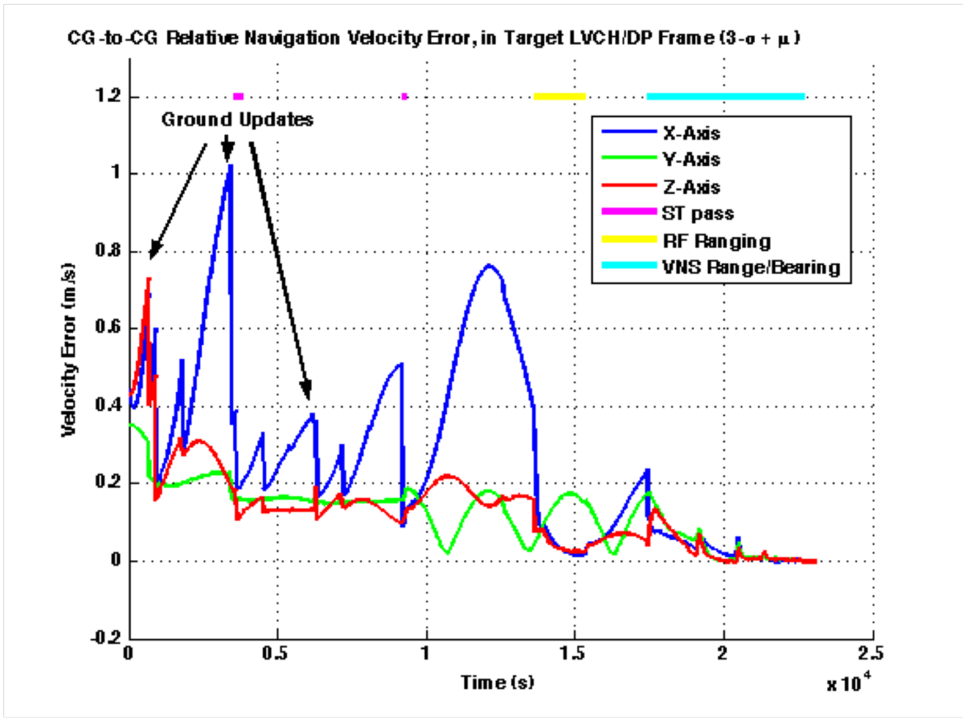

Figure 2 Relative velocity errors

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