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DEALING WITH UNCERTAINTIES IN INITIAL ORBIT DETERMINATION

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A method to deal with uncertainties in initial orbit determination (IOD) is presented. This is based on the use of Taylor differential algebra (DA) to nonlinearly map the observation uncertainties from the observation space to the state space. When a minimum set of observations is available DA is used to expand the solution of the IOD problem in Taylor series with respect to measurement errors. When more observations are available high order inversion tools are exploited to obtain full state pseudo-observations at a common epoch. The mean and covariance of these pseudo-observations are nonlinearly computed by evaluating the expectation of high order Taylor polynomials. Finally, a linear updating scheme is employed to update the current knowledge of the orbit. Angles-only observation are considered and simplified Keplerian dynamics adopted to ease the explanation. Four test cases of orbit determination of artificial satellites in different orbital regimes are presented to discuss the feature and performances of the proposed methodology.

INTRODUCTION

Orbit determination is typically divided into two phases. When the number of observation is equal to the number of unknowns a nonlinear system of equations need to be solved. This problem is known as initial (or preliminary) orbit determination (IOD). When more observations are available accurate orbit determination can be performed. IOD typically delivers a single solution (or a limited number of solutions) that exactly produces the available observations. In addition, in IOD simplified dynamical models are often used (e.g. Keplerian motion) and measurement errors are not taken into account (the problem is deterministic). When more observations are available the approach is commonly treated as stochastic by considering the observations' noise. This problem is usually solved as an optimization, in which the (optimal) solution minimizes the observation residuals. The solution is commonly obtained via batch estimation, e.g. weighted linearized least squares, or a sequential estimation, e.g. extended Kalman Filtering.[1]

The focus of this paper is the orbit determination of resident space objects (RSO) observed on a single passage with optical sensors. Thus, the problem is angles-only orbit determination. In order to determine the orbit IOD is solved to provide an initial estimate followed by a procedure to update the initial solution based on the additional observations.

Angles-only IOD is the subject of much research. Gauss'[2] and Laplace's[3] methods are commonly used to determine a Keplerian orbit that fits three astrometric observations. These methods have

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been revisited and analyzed by a large number of authors 4, 5, 6. New methods have also been introduced such as the Double r-iteration technique of Escobal[7] and the approach of Gooding.[8]

In 2012 Armellin at al.[9] proposed a IOD solver based on the solution of a Lambert's problem (between the second and the third observations) and a Kepler's problem between the first and second observation. The method iterates on the slant ranges at the second and third observations in order to drive to zero the observational defects at the first observation. The iterations were carried out with a high-order extension of Newton's method enabled by differential algebra (DA). In addition, high order Taylor expansions were exploited to nonlinearly map the uncertainties from the observation space to the state space.

In this work a modified version of the method is proposed, in which all the three slant ranges as unknowns. The approach is based on the solution of two Lambert's problems and using the continuity of the velocity vector at the central observation as constraint. The method has no restrictions on the geometry of the observations and it can deal with both short and long gaps. As in the previous work the solution is obtained with a high-order Newton's iteration scheme enabled by DA. This approach allows the algorithm to both convergence in few iterations and map uncertainties form the observation space to the state space. Thus, the initial orbit is provided with accurate and nonlinear statistical information.

When multiple observations on the same passage are available the IOD solution is updated. Instead of adopting a classical least squares approach (which employs the linearization of the dynamics and of the measurement functions[10]) high order inversion tools available in DA are exploited to nonlinearly map group of observations to the state space at a common epoch, thus producing full state pseudo-observations. The mean and covariance of these pseudo-observations are nonlinearly computed by evaluating the expectation of the related high order Taylor polynomials. Finally, a linear updating scheme is utilized to update the current knowledge of the state mean and covariance.

The paper is organized as follows. A brief introduction on the DA tools used for the implementation of the algorithm is given first. This covers the methods to expand the solution of ordinary differential equations (ODE), compute the expansion of the solution of implicit parametric equations, and the algorithm to map statistics through nonlinear transformations. The following sections describes the main algorithms developed in this work, i.e. the angles-only IOD solver and the updating scheme. Simulated observational scenarios for a Low Earth Orbit (LEO), a Geosynchronous Transfer Orbit (GTO), a Geosynchronous Orbit (GEO) and a Molniya are used to analyzed the performances of the implemented methods. Some final remarks conclude the paper.

DIFFERENTIAL ALGEBRA TOOLS

DA supplies the tools to compute the derivatives of functions within a computer environment.[11] More specifically, by substituting the classical implementation of real algebra with the implementation of a new algebra of Taylor polynomials, any function f of v variables is expanded into its Taylor polynomial up to an arbitrary order n with limited computational effort. In addition to basic algebraic operations, operations for differentiation and integration can be easily introduced in the algebra, thusly finalizing the definition of the differential algebra structure of DA.[12,13] Similarly to algorithms for floating point arithmetic, various DA algorithms were introduced, including methods to perform composition of functions, to invert them, to solve nonlinear systems explicitly, and to treat common elementary functions.[14] The differential algebra used for the computations in this work was implemented in the software COSY INFINITY.[15] The reader may refer to Di Lizia et al.[16] for the DA notation adopted throughout the paper.

High-order expansion of the solution of ODE

An important application of DA is the automatic high order expansion of the solution of an ODE in terms of the initial conditions.[14,16] This can be achieved by replacing the operations in a classical numerical integration scheme, including evaluation of the right hand side, by the corresponding DA operations. This way, starting from the DA representation of an initial condition x_0, DA ODE integration allows the propagation of the Taylor expansion of the flow in x_0 forward in time, up to any final time t_f. Any explicit ODE integration scheme can be rewritten as a DA integration scheme in a straight-forward way. For the numerical integrations presented in this paper, a DA version of a 7/8 Dormand-Prince (8-th order solution for propagation, 7-th order solution for step size control) Runge-Kutta scheme is used. The main advantage of the DA-based approach is that there is no need to write and integrate variational equations in order to obtain high order expansions of the flow. It is therefore independent of the particular right hand side of the ODE and the method is quite efficient in terms of computational cost.

Expansion of the solution of parametric implicit equations

Well-established numerical techniques (e.g., Newton's method) exist, which can effectively identify the solution of a classical implicit equation

$$\boldsymbol{f}(\boldsymbol{x}) = 0 \tag{1}$$

with $\boldsymbol{f} : \Re^n \to \Re^n$. Suppose an explicit dependence on a vector of parameters $\boldsymbol{p}$ can be highlighted in the previous vector function $\boldsymbol{f}$, which leads to the parametric implicit equation

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) = 0. \tag{2}$$

Suppose the previous equation is to be solved, whose solution is represented by the function $\boldsymbol{x}(\boldsymbol{p})$ returning the value of $\boldsymbol{x}$ solving Eq. (2) for any value of $\boldsymbol{p}$. Thus, the dependence of the solution of the implicit equation on $\boldsymbol{p}$ is of interest. DA techniques can effectively handle the previous problem by identifying the function $\boldsymbol{x}(\boldsymbol{p})$ in terms of its Taylor expansion with respect to $\boldsymbol{p}$. This result is achieved by applying partial inversion techniques as detailed in Ref. 16.

The final result is

$$[\boldsymbol{x}] = \boldsymbol{x} + \mathcal{M}_{\boldsymbol{x}}(\delta \boldsymbol{p}), \tag{3}$$

which is the k-th order Taylor expansion of the solution of the implicit equation. For every value of $\boldsymbol{\delta p}$, the approximate solution of $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) = 0$ can be easily computed by evaluating the Taylor polynomial (3). The solution obtained by means of Eq. (3) is a Taylor approximation of the exact solution of Eq. (2). The accuracy of the approximation depends on both the order of the Taylor expansion and the displacement $\delta \boldsymbol{p}$ from the reference value of the parameter.

Nonlinear mapping of the estimate statistics

Consider a random vector $\boldsymbol{x} \in \Re^n$ with probability density function $p(\boldsymbol{x})$ and a second random vector $\boldsymbol{y} \in \Re^m$ related to $\boldsymbol{x}$ through the nonlinear transformation

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}). \tag{4}$$

The goal is to calculate a consistent estimate of the main cumulants of the transformed probability density function $p(\boldsymbol{y})$. Since $\boldsymbol{f}$ is a generic nonlinear function this formulation includes a wide range of problems involving uncertainty propagation (uncertainty propagation through nonlinear dynamics, uncertainty propagation through nonlinear coordinate transformations, etc.).

The Taylor expansion of $\boldsymbol{y}$ with respect to deviations $\boldsymbol{\delta} x$ can be obtained automatically by initializing the independent variable as a DA variable and evaluating Eq. (4) in the DA framework. This procedure delivers

$$[\boldsymbol{y}] = \boldsymbol{f}([\boldsymbol{x}]) = \boldsymbol{y} + \mathcal{M}_{\boldsymbol{y}}(\delta \boldsymbol{x}) = \sum_{p_1+\cdots+p_n \leq k} \boldsymbol{c}_{p_1 \ldots p_n} \cdot \delta x_1^{p_1} \cdots \delta x_n^{p_n}, \tag{5}$$

where in this expression $\boldsymbol{y}$ is the zeroth order term of the expansion map, and $\boldsymbol{c}_{p_1 \ldots p_n}$ are the Taylor coefficients of the resulting Taylor polynomial

$$\boldsymbol{c}_{p_1 \ldots p_n} = \frac{1}{p_1! \cdots p_n!} \cdot \frac{\partial^{p_1+\cdots+p_n} \boldsymbol{f}}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}}. \tag{6}$$

The evaluation of Eq. (5) for a selected value of $\delta \boldsymbol{x}$ supplies the k-th order Taylor approximation of $\boldsymbol{y}$ corresponding to the displaced independent variable. Of course, the accuracy of the expansion map is a function of the expansion order and can be controlled by tuning it.

The Taylor series in the form of Eq. (5) can be used to efficiently compute the propagated statistics.[17,18] The method consists in analytically describing the statistics of the solution by computing the l-th moment of the transformed pdf using a proper form of the l-th power of the solution Map in Eq. (5).

For a generic scalar random variable x with pdf $p(x)$ the first four moments can be written as

$$\begin{cases} \mu = E\{x\} \\ P = E\{(x-\mu)^2\} \\ \gamma = \dfrac{E\{(x-\mu)^3\}}{\sigma^3} \\ \kappa = \dfrac{E\{(x-\mu)^4\}}{\sigma^4} - 3, \end{cases} \tag{7}$$

where μ is the mean value, P is the covariance, γ and κ are the skewness and the kurtosis, respectively,[19] and the expectation value of x is defined as

$$E\{x\} = \int_{-\infty}^{+\infty} x p(x) dx. \tag{8}$$

The moments of the transformed pdf in (4) can be computed by applying the multivariate form of Eq. (7) to the Taylor expansion (5). The result for the first two moments becomes

$$\begin{cases} \boldsymbol{\mu}_{\boldsymbol{y}_i} = E\{[\boldsymbol{y}_i]\} = \displaystyle\sum_{p_1+\cdots+p_n \leq k} \boldsymbol{c}_{i,p_1 \ldots p_n} E\{\delta x_1^{p_1} \cdots \delta x_n^{p_n}\} \\ \boldsymbol{P}_{\boldsymbol{y}_i \boldsymbol{y}_j} = E\{([\boldsymbol{y}_i] - \boldsymbol{\mu}_i)([\boldsymbol{y}_j] - \boldsymbol{\mu}_j)\} = \displaystyle\sum_{\substack{p_1+\cdots+p_n \leq k, \\ q_1+\cdots+q_n \leq k}} \boldsymbol{c}_{i,p_1 \ldots p_n} \boldsymbol{c}_{j,q_1 \ldots q_n} E\{\delta x_1^{p_1+q_1} \cdots \delta x_n^{p_n+q_n}\}, \end{cases} \tag{9}$$

where $\boldsymbol{c}_{i,p_1\ldots p_n}$ are the Taylor coefficients of the Taylor polynomial describing the i-th component of $[\boldsymbol{y}]$. Note that in the covariance matrix formula the coefficients $\boldsymbol{c}_{i,p_1\ldots p_n}$ and $\boldsymbol{c}_{j,q_1\ldots q_n}$ are updated to include the subtraction of the mean. The coefficients of the higher order moments are computed by implementing the required operations (e.g. $([\boldsymbol{y}_i]-\boldsymbol{\mu}_i)([\boldsymbol{y}_j]-\boldsymbol{\mu}_j)$ for the second order moment) on Taylor polynomials in the DA framework. The expectation values on the right side of Eq. (9) are function of $p(\boldsymbol{x})$. It follows that if the initial distribution is known, all of the moments of the transformed pdf $p(\boldsymbol{y})$ can be calculated. The number of monomials for which it is necessary to compute the expectation increases with the order of the Taylor expansion and, of course, with the order of the calculated moment. Note that, at this time, no hypothesis on the initial pdf has been made. Thus, the method can be applied independently of the distribution.

Under the assumption $\boldsymbol{x}$ is a Gaussian random variable (GRV), $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{P})$, $\boldsymbol{\mu}$ is the mean vector and $\boldsymbol{P}$ the covariance matrix. An important property of Gaussian distributions is that the statistics of a GRV can be completely described by the first two moments. In case of zero mean, the expression for computing higher-order moments in terms of the covariance matrix is due to Isserlis.[20] In physics literature, Isserlis's formula is known as the Wick's formula.

Let s_1 to s_n be nonnegative integers, and $s = s_1 + s_2 + \cdots + s_n$. Then the Wick's formula states that

$$E\{x_1^{s_1} x_2^{s_2} \ldots x_n^{s_n}\} = \begin{cases} 0, & \text{if } s \text{ is odd} \\ Haf(\boldsymbol{P}), & \text{if } s \text{ is even} \end{cases} \tag{10}$$

where $Haf(\boldsymbol{P})$ is the hafnian of $P = (\sigma_{ij})$, which is defined as

$$Haf(\boldsymbol{P}) = \sum_{p \in \prod_s} \prod_{i=1}^{\frac{s}{2}} \sigma_{p_{2i-1}, p_{2i}}, \tag{11}$$

and $\prod_s$ is the set of all permutations p of $\{1, 2, \ldots, s\}$ satisfying the property $p_1 < p_3 < p_5 < \ldots < p_{s-1}$ and $p_1 < p_2,\ p_3 < p_4, \ldots, p_{s-1} < p_s$.[21]

We observe that the expectation value terms of Eq. (9) can be computed using Eq. (10), and the resulting moments can be used to describe the transformed pdf.

DA-BASED ANGLES-ONLY IOD

In the classical angles-only IOD problem there are three optical observation at epochs t_i, with $i = 1, \ldots, 3$. The observations consist in three couples of right ascension and declinations, (α_i, δ_i). These observations provides three inertial line-of-sight vectors $\hat{\boldsymbol{\rho}}_i$, i.e. the unit vectors from the observer's location (on the Earth's surface) to the observed object.

Assume to have initial guessed values of the slant ranges ρ_i, or equivalently for the orbit radii r_i (e.g. from the solution Gauss' 8-th degree polynomial). This section presents a high order iterative procedure with the following objectives: a) find the exact values of ρ_i assuming Keplerian dynamics, and b) express the functional dependence of the solution of the IOD problem with respect to observation uncertainties in terms of a high-order Taylor polynomial.

Start by initializing the observations as DA variables

$$\begin{aligned} [\boldsymbol{\alpha}] &= \boldsymbol{\alpha} + \delta\boldsymbol{\alpha} \\ [\boldsymbol{\delta}] &= \boldsymbol{\delta} + \delta\boldsymbol{\delta}, \end{aligned} \tag{12}$$

in which the observations are grouped in two homogeneous vectors, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)$, and $\delta\boldsymbol{\alpha}$ and $\delta\boldsymbol{\delta}$ account for measurement uncertainties. The line of sight vectors at t_1, t_2 and t_3 become

$$\begin{aligned} [\hat{\boldsymbol{\rho}}_1] &= \hat{\boldsymbol{\rho}}_1 + \mathcal{M}_{\hat{\boldsymbol{\rho}}_1}(\delta\alpha_1, \delta\delta_1) \\ [\hat{\boldsymbol{\rho}}_2] &= \hat{\boldsymbol{\rho}}_2 + \mathcal{M}_{\hat{\boldsymbol{\rho}}_2}(\delta\alpha_2, \delta\delta_2) \\ [\hat{\boldsymbol{\rho}}_3] &= \hat{\boldsymbol{\rho}}_3 + \mathcal{M}_{\hat{\boldsymbol{\rho}}_3}(\delta\alpha_3, \delta\delta_3), \end{aligned} \tag{13}$$

where $\mathcal{M}_{\hat{\boldsymbol{\rho}}_i}$ is an arbitrary order Taylor polynomial that describes the effect of an observation uncertainty on the line of sight.

Similarly, the topocentric distances at t_1, t_2 and t_3 are described as DA variables

$$\begin{aligned} [\rho_1]^{1^-} &= \rho_1^{1^-} + \delta\rho_1 \\ [\rho_2]^{1^-} &= \rho_2^{1^-} + \delta\rho_2 \\ [\rho_3]^{1^-} &= \rho_3^{1^-} + \delta\rho_3, \end{aligned} \tag{14}$$

or in more compact form

$$[\boldsymbol{\rho}]^{1^-} = \boldsymbol{\rho}^{1^-} + \delta\boldsymbol{\rho}, \tag{15}$$

where the superscript 1^- indicates the first step of the iterative procedure, and $\rho_1^{1^-}$, $\rho_2^{1^-}$, and $\rho_3^{1^-}$ are the initial guessed values for the slant ranges.

The spacecraft position vectors can be written (by summing the known observer's locations) as

$$\begin{aligned} [\boldsymbol{r}_1] &= \boldsymbol{r}_1 + \mathcal{M}_{\boldsymbol{r}_1}(\delta\alpha_1, \delta\delta_1, \delta\rho_1) \\ [\boldsymbol{r}_2] &= \boldsymbol{r}_2 + \mathcal{M}_{\boldsymbol{r}_2}(\delta\alpha_2, \delta\delta_2, \delta\rho_2) \\ [\boldsymbol{r}_3] &= \boldsymbol{r}_3 + \mathcal{M}_{\boldsymbol{r}_3}(\delta\alpha_3, \delta\delta_3, \delta\rho_3). \end{aligned} \tag{16}$$

A DA-based Lambert's problem[22] can be solved between with $[\boldsymbol{r}_1]$ and $[\boldsymbol{r}_2]$, and between $[\boldsymbol{r}_2]$ and $[\boldsymbol{r}_3]$. Using the DA-implementation of Lambert's problem we obtain two Taylor polynomial approximation for the velocity vectors at t_2

$$\begin{aligned} [\boldsymbol{v}_2^-] &= \boldsymbol{v}_2^- + \mathcal{M}_{\boldsymbol{v}_2^-}(\delta\alpha_1, \delta\delta_1, \delta\alpha_2, \delta\delta_2, \delta\rho_1, \delta\rho_2) \\ [\boldsymbol{v}_2^+] &= \boldsymbol{v}_2^+ + \mathcal{M}_{\boldsymbol{v}_2^+}(\delta\alpha_2, \delta\delta_2, \delta\alpha_3, \delta\delta_3, \delta\rho_2, \delta\rho_3) \end{aligned} \tag{17}$$

Note that the two above expressions are different for two reasons. Firstly we start from values of the slant ranges that are not the solution of the problem. Secondly the they have different functional dependence on the angles. The goal is to find the nominal values of the slant ranges such that the velocity vector is continuos at the midpoint, i.e. we want to find the exact values of ρ_1, ρ_2, and ρ_3 and the high-order Taylor expansion of spacecraft state at t_2 with respect to observation uncertainties. We start by defining the Taylor map of the defects

$$[\Delta\tilde{\boldsymbol{v}}_2] = [\boldsymbol{v}_2^+] - [\boldsymbol{v}_2^+] = \Delta\tilde{\boldsymbol{v}}_2 + \mathcal{M}_{\Delta\tilde{\boldsymbol{v}}_2}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}, \delta\boldsymbol{\rho}). \tag{18}$$

Note that for the exact values of ρ_1, ρ_2 and ρ_3 the constant part of maps in Eq. (18) is zero. We now need to find the variations $\delta\boldsymbol{\rho}$ necessary to cancel out these constants and to express $\boldsymbol{r}_2$ and $\boldsymbol{v}_2$ as Taylor polynomials in $\delta\boldsymbol{\alpha}$ and $\delta\boldsymbol{\delta}$ only.

The first step is to work with an origin preserving map

$$[\Delta\boldsymbol{v}_2] = [\Delta\tilde{\boldsymbol{v}}_2] - \Delta\tilde{\boldsymbol{v}}_2 = \mathcal{M}_{\Delta\boldsymbol{v}_2}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}, \delta\boldsymbol{\rho}) \tag{19}$$

and to build an augmented Taylor polynomial by adding identities in observation deltas

$$\begin{bmatrix} \Delta \boldsymbol{v}_2 \\ \delta \boldsymbol{\alpha} \\ \delta \boldsymbol{\delta} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\Delta \boldsymbol{v}_2} \\ \mathcal{I}_{\boldsymbol{\alpha}} \\ \mathcal{I}_{\boldsymbol{\delta}} \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{\alpha} \\ \delta \boldsymbol{\delta} \\ \delta \boldsymbol{\rho} \end{bmatrix}. \tag{20}$$

This polynomial map can be inverted using ad-hoc algorithms implemented in COSY-Infinity, yielding

$$\begin{bmatrix} \delta \boldsymbol{\alpha} \\ \delta \boldsymbol{\delta} \\ \delta \boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\Delta \boldsymbol{v}_2} \\ \mathcal{I}_{\boldsymbol{\alpha}} \\ \mathcal{I}_{\boldsymbol{\delta}} \end{bmatrix}^{-1} \begin{bmatrix} \Delta \boldsymbol{v}_2 \\ \delta \boldsymbol{\alpha} \\ \delta \boldsymbol{\delta} \end{bmatrix}. \tag{21}$$

Extracting the three last lines we obtain

$$\begin{bmatrix} \delta \boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\boldsymbol{\rho}} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{v}_2 \\ \delta \boldsymbol{\alpha} \\ \delta \boldsymbol{\delta} \end{bmatrix}. \tag{22}$$

We now evaluate the map (22) in $[\Delta \boldsymbol{v}_2] = -\Delta \tilde{\boldsymbol{v}}_2$, obtaining

$$[\boldsymbol{\rho}]^{1^+} = \boldsymbol{\rho}^{1^+} + \mathcal{M}_{\boldsymbol{\rho}}(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta}) \tag{23}$$

where the subscript 1^+ indicates the Taylor polynomial of the corrections of the topocentric distances to be applied at the end of the first iteration. Note that this step is the high-order counterpart of classical Newton's method.

The second iteration starts with the Taylor polynomials of the topocentric distances given by

$$[\boldsymbol{\rho}]^{2^-} = [\boldsymbol{\rho}]^{1^-} + [\boldsymbol{\rho}]^{1^+} + \delta \boldsymbol{\rho} = \boldsymbol{\rho}^{2^-} + \mathcal{M}_{\boldsymbol{\rho}}(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta}, \delta \boldsymbol{\rho}) \tag{24}$$

where now the explicit dependence on the entire set of observables appears. Thus, from the second iteration, the Taylor polynomials (16)–(17) depend on all $(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta}, \delta \boldsymbol{\rho})$. The iterative procedure ends when the values of $\Delta \tilde{\boldsymbol{v}}_2$ are smaller than a prescribed tolerance. The Taylor polynomials of the topocentric distances at the last iteration k are

$$[\boldsymbol{\rho}] = [\boldsymbol{\rho}]^{k^-} + [\boldsymbol{\rho}]^{k^+} = \boldsymbol{\rho} + \mathcal{M}_{\boldsymbol{\rho}}(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta}) \tag{25}$$

Using these expressions the spacecraft position and velocity vectors at t_2 assume the form

$$\begin{aligned} [\boldsymbol{r}_2] &= \boldsymbol{r}_2 + \mathcal{M}_{\boldsymbol{r}_2}(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta}) \\ [\boldsymbol{v}_2] &= \boldsymbol{v}_2 + \mathcal{M}_{\boldsymbol{v}_2}(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta}). \end{aligned} \tag{26}$$

or more compactly

$$[\boldsymbol{x}_2] = \boldsymbol{x}_2 + \mathcal{M}_{\boldsymbol{x}_2}(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta}), \tag{27}$$

where $\boldsymbol{x}_2 = (\boldsymbol{r}_2, \boldsymbol{v}_2)$.

Note that, as a result of the iterative procedure, $\boldsymbol{r}_2$ and $\boldsymbol{v}_2$ exactly satisfy (in the two-body model) the nominal observation set $(\boldsymbol{\alpha}, \boldsymbol{\delta})$. Furthermore, for any displaced value of the observables, the solution of the preliminary determination problem is computed by evaluating the polynomial (26) in the corresponding values of $(\delta \boldsymbol{\alpha}, \delta \boldsymbol{\delta})$. Map (27) is an arbitrary order Taylor polynomial in $\delta \boldsymbol{\alpha}$ and $\delta \boldsymbol{\delta}$, which maps the uncertainties from the observable space to the spacecraft state phase. In particular, using the approach described in Section "Nonlinear mapping of the estimate statistics" we can compute the statistical moments of $\boldsymbol{x}$, given the statistics of the measurements.

DA-INVERSION IOD

When more than three optical observations are available the solution (reference state and associated statistics) of the IOD problem needs to be updated to include the additional information. This is carried out through a high-order filtering technique based on nonlinear mapping of statistics and linear update scheme, in which only the pdf of the measurements is constrained to be Gaussian.

The optimal linear estimate of a state $\boldsymbol{x}$ based on a measurement $\boldsymbol{y}$ is given by

$$\hat{\boldsymbol{x}} = \boldsymbol{\mu}_{\boldsymbol{x}} + \boldsymbol{P}_{\boldsymbol{xy}} \boldsymbol{P}_{\boldsymbol{yy}}^{-1} (\tilde{\boldsymbol{y}} - \boldsymbol{\mu}_{\boldsymbol{y}}) \tag{28}$$

where $\boldsymbol{\mu}_{\boldsymbol{x}}$ is the state mean, $\boldsymbol{P}_{\boldsymbol{xy}}$ is the joint covariance of the state and the measurement, and $\boldsymbol{P}_{\boldsymbol{yy}}$ is the covariance of the measurement. For a general non-linear measurement with additive noise $\tilde{\boldsymbol{y}} = \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{\eta}$, calculating $\boldsymbol{\mu}_{\boldsymbol{y}}$ and the covariance matrices requires full knowledge of the distribution of the state. This requirement has two consequences: first it means that the state and its uncertainty need to be propagated forward to the measurement time, and second that statistics of the measurement need to be calculated through a nonlinear transformation of the current state. In this work we propose addressing this issue in a different way. The state is always estimated at a fixed epoch time, and the nonlinear map to transport it to any other epoch is calculated with the DA framework. Instead of working with $\boldsymbol{y}$ as a function of $\boldsymbol{x}$, a full pseudo-measurement of the state is generated from $\boldsymbol{y}$; the inverse of the non-linear map from the state to the measurement is readily available from COSY-Infinity. The advantage of this approach is that only the distribution of the measurement noise is assumed Gaussian while the distribution of the state is left unconstrained.

Consider a time span $[t_0, t_f]$ and let $\boldsymbol{x}_k$ be the state variable at some time $t_k \in [t_0, t_f]$. Consider also a set of N measurements $\tilde{\boldsymbol{y}}_i$ given at times $t_i \in [t_0, t_f]$ with $i = 1, \ldots, N$. Given the current estimate of the state $\boldsymbol{\mu}_{\boldsymbol{x}_k}^-$ and the related error statistics, we can always define the estimated state as a DA variable and compute the predicted measurement at t_i in the DA framework. The relation between state and measurement is a nonlinear map that accounts for the forward propagation of the initial condition and the measurement function. Under proper conditions this relation can be inverted to map the observation space at t_i into the state space at t_k. The main cumulants of the resulting map can be computed as described in the previous section, with the assumption that the statistics of the measurement errors is Gaussian. The computed mean and covariance are exploited to update the knowledge of $\boldsymbol{x}_k$ using a linear update scheme. This can be done for groups of measurements for which the dimension of measurement vector $\boldsymbol{y}_i$ is equal to the dimension of the state vector, and the map is invertible.

The resulting method can be made recursive and summarized as follows. From the IOD algorithm we start from an initial value of the state estimate and covariance, $\hat{\boldsymbol{x}}_k^- = \boldsymbol{\mu}_{\boldsymbol{x}_k}^-$ and $P_{\boldsymbol{x}_k \boldsymbol{x}_k}^-$ (in general $t_k = t_2$, the epoch of the central observation in the IOD problem.) Define the current estimate at time of interest t_k as a DA variable; i.e.,

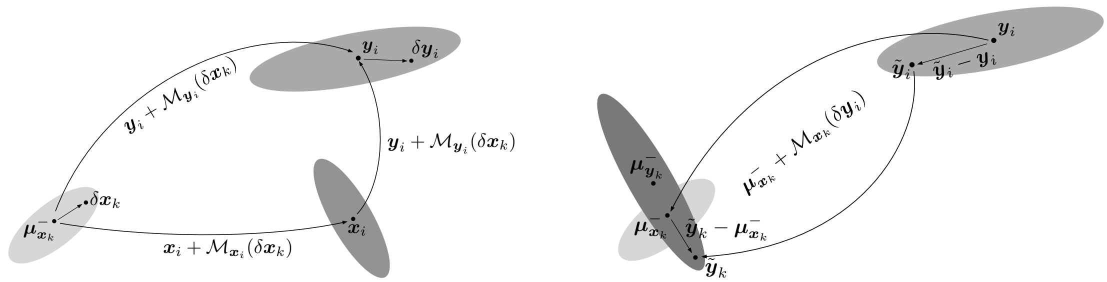
$$[\boldsymbol{x}_k] = \hat{\boldsymbol{x}}_k^- + \delta \boldsymbol{x}_k. \tag{29}$$

and propagate it to time t_i when a measurement becomes available. The result assumes the form of the following high-order Taylor expansion map

$$[\boldsymbol{x}_i] = \hat{\boldsymbol{x}}_i + \mathcal{M}_{\boldsymbol{x}_i}(\delta \boldsymbol{x}_k). \tag{30}$$

Then, use the measurement equation to compute

$$[\boldsymbol{y}_i] = \boldsymbol{h}([\boldsymbol{x}_i]) = \hat{\boldsymbol{y}}_i + \mathcal{M}_{\boldsymbol{y}_i}(\delta \boldsymbol{x}_k), \tag{31}$$



(a) Direct maps representation (b) Inverse map representation

Figure 1: Sketch of the Taylor maps involved in the construction of the DA-base map inversion nonlinear filter.

where $\boldsymbol{h}$ represents the measurement function. Figure 1(a) can be used by the reader to better understand the meaning of Maps (30)–(31).

The next step consists in defining an origin preserving map

$$\delta\boldsymbol{y}_i = [\boldsymbol{y}_i] - \hat{\boldsymbol{y}}_i = \mathcal{M}_{\delta\boldsymbol{y}_i}(\delta\boldsymbol{x}_k). \tag{32}$$

This polynomial map can be inverted if two conditions are satisfied: the map must be square and all the measurements must be independent. If these requirements are satisfied, we can invert Map (32) using algorithms implemented in COSY-Infinity, obtaining

$$\delta\boldsymbol{x}_k = \mathcal{M}_{\delta\boldsymbol{x}_k}(\delta\boldsymbol{y}_i). \tag{33}$$

We now substitute in Map (29) the expression of $\delta\boldsymbol{x}_k$ from (33), yielding

$$[\boldsymbol{x}_k] = \hat{\boldsymbol{x}}_k^- + \mathcal{M}_{\boldsymbol{x}_k}(\delta\boldsymbol{y}_i). \tag{34}$$

Note that this map now represents the pseudo-measurement of the state $\boldsymbol{x}_k$ based on the observation $\tilde{\boldsymbol{y}}_i$, so it is renamed as

$$[\boldsymbol{z}_k] = \hat{\boldsymbol{x}}_k^- + \mathcal{M}_{\boldsymbol{y}_k}(\delta\boldsymbol{y}_i). \tag{35}$$

By construction the constant part of Eq. (35) is equal to the state estimate at step k, i.e. $\hat{\boldsymbol{x}}_k^-$, but its statistical moments are different to those of $\boldsymbol{x}_k$, due to the nonlinear contribution of $\mathcal{M}_{\boldsymbol{x}_k}(\delta\boldsymbol{y}_i)$ (as highlighted in Fig. 1(b)). We can now apply Eq. (9) to Taylor expansion (35) to compute the statistics of the random variable $\boldsymbol{z}_k$ and, in particular, the first two moments $\boldsymbol{\mu}_{\boldsymbol{z}_k}$ and $\boldsymbol{P}_{\boldsymbol{z}_k\boldsymbol{z}_k}$. The computed mean can be treated as the "predicted measure" of the state at time t_k, with measurement error defined by $\boldsymbol{P}_{\boldsymbol{z}_k\boldsymbol{z}_k}$. Thus, we can update the initial estimate and error covariance, using the least square method. This can be done using the Kalman filter update equations that, applied to the current problem, read

$$\boldsymbol{K} = \boldsymbol{P}^-_{\boldsymbol{x}_k\boldsymbol{x}_k}\left(\boldsymbol{P}^-_{\boldsymbol{x}_k\boldsymbol{x}_k} + \boldsymbol{P}_{\boldsymbol{z}_k\boldsymbol{z}_k}\right)^{-1}, \tag{36}$$

$$\hat{\boldsymbol{x}}_k^+ = \hat{\boldsymbol{x}}_k^- + \boldsymbol{K}\left(\tilde{\boldsymbol{z}}_k - \boldsymbol{\mu}_{\boldsymbol{z}_k}\right), \tag{37}$$

$$\boldsymbol{P}^+_{\boldsymbol{x}_k\boldsymbol{x}_k} = (\boldsymbol{I} - \boldsymbol{K})\,\boldsymbol{P}^-_{\boldsymbol{x}_k\boldsymbol{x}_k}(\boldsymbol{I} - \boldsymbol{K})^T + \boldsymbol{K}\boldsymbol{P}_{\boldsymbol{z}_k\boldsymbol{z}_k}\boldsymbol{K}^T, \tag{38}$$

where $\boldsymbol{\mu}^+_{\boldsymbol{x}_k}$ the updated estimate at time t_k and $\boldsymbol{P}^+_{\boldsymbol{x}_k\boldsymbol{x}_k}$ the updated estimation error covariance matrix. When another measurement becomes available, we can define the state at time t_k as a new DA variable, centered in the new estimate $\boldsymbol{\mu}^+_{\boldsymbol{x}_k}$, and iterate the process. Note that $\tilde{\boldsymbol{z}}_k$ is the true measurement at t_i mapped to the state space at time t_k, which is readily available by evaluating Map (35) for $\delta \boldsymbol{y}_i = \tilde{\boldsymbol{y}}_i - \boldsymbol{y}_i$.

We said that Map (32) must be square in order to be invertible. It follows that if the measurement vector has smaller dimension than the state vector, after the first measurement is received we can not proceed with the update, but we have to wait for additional measurements (i.e. in the optical case three observations are needed). When the number of independent scalar measurements equals the dimension of the state variable, we can define an augmented measurement vector that can be used to build Maps (31) and (32).

Once the final estimate of the state at time t_k is obtained, the statistics of the solution can be computed at any time via propagation and DA-based expectation evaluation.

TEST CASES

In the following section, the algorithms for IOD are run considering single-pass optical observations of four objects as listed in Table 1.

Table 1: Test cases: orbital parameters

Test Case		A	B	C	D
Orbit type		LEO	GEO	GTO	Molniya
NORAD ID		04784	26824	23238	40296
Epoch	JED	2457155.973681	2457163.282443	2457167.100821	2457165.070824
a	km	7353.500	42143.7813	26569.834	26569.833
e	–	0.0026401	0.0002262	0.7233923	0.72339221
i	deg	74.0295	0.03570	62.79393	62.79393
Ω	deg	179.64010	26.27830	344.537891	344.53789
ω	deg	359.07890	42.05210	271.34770	271.34770
M	deg	99.45760	72.45500	347.72640	347.72640

The observations are all simulated from Teide Observatory, Tenerife, Canary Islands, Spain (observation code 954). The simulation windows are summarized in Table 2. For all the cases 15 equally spaced optical observations are simulated within the observation window. The spacecraft is considered observable when it has an elevation above 10 deg, it is in sunlight, and the Sun has an elevation lower than -7 deg). As a result, different gaps between observations are considered, going from a separation of 43.2 s for the LEO case to 2160 s for the GEO case. This approximately translates in a average angular separation between observations (on the orbit) of 2.4 deg the LEO and 9.1 deg for GEO case. The GTO object is observed before the apogee on an orbital arc of approximately 20.7 deg. The average separation between observations is 1.5 deg, with maximum and minimum values of 1.9 and 1.3 deg, respectively. On the contrary the Molniya object is observed before the apogee on an orbital arc of 13.4 deg. In this case the mean, maximum, and minimum observation separations are 1, 1.1, and 0.8 deg. For all the cases the central observations (i.e. observations 7,8,9) are used for the IOD, whereas the remaining ones for orbit update. Finally, pertaining to the observation accuracies we consider a standard deviation of 0.5 arcsec for all the observations, except for the LEO case for which the error is increased by one order of magnitude to account for the faster motion of the object.

Table 2: Test cases: observation windows

Test Case	Observation Window						Δt	$\sigma_{\alpha,\delta}$
	yr	mo	day_0	day_N	hr_0	hr_N	hr	arcsec
A	2015	MAY	15	15	22.250	22.418	0.012	5
B	2015	MAY	22	23	21.000	05.400	0.600	0.5
C	2015	JUN	02	02	03.550	05.580	0.145	0.5
D	2015	MAY	22	22	20.600	23.400	0.200	0.5

CONCLUSIONS

In this paper the problem of nonlinear filtering has been addressed. Working in the differential algebra framework we derived a high-order filter, called differential algebra-based map inversion filter. This filtering algorithm is based on nonlinear mapping of statistics and linear update scheme, in which only the probability density function of the measurements' error is constrained to be Gaussian. No hypothesis on the state probability density function is made. The proposed filter is compared to the conventional extended Kalman filter and to the unscented Kalman filter in a Earth-orbiting spacecraft application. The filter simulations are carried out assuming the dynamics of the system are perfectly known and given by the two-body problem, but there are random errors in the initial state and in the measurements. The results show that the proposed filter provides better performance that the linearized solution and, in some cases (i.e., when the initial uncertainty, the measurement noise, and/or the filtering time step are large) also than the unscented Kalman filter.

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