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RECURSIVE UPDATE FILTER APPLIED TO SPACECRAFT RENDEZVOUS

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Nonlinear filters are often very computationally expensive and usually not suitable for real-time applications. Real-time navigation algorithms are typically based on linear estimators, such as the extended Kalman filter (EKF). In a typical spacecraft rendezvous very accurate lidar measurements are available to navigate the chaser vehicle with respect to the target. Incorporating these highly accurate nonlinear measurements can cause filter divergence in the presence of highly uncertain initial conditions. This work proposes to use a nonlinear estimator to accurately navigate the chaser vehicle to the target without the need of any *ad hoc* algorithm to keep the EKF consistent.

INTRODUCTION

The well known Kalman filter [1, 2] is an optimal estimation algorithm. The optimality holds in terms of minimum mean square error and maximum likelihood estimation under several conditions. These conditions are met when all noises and the initial estimation error are Gaussian, and when the dynamics and measurements are linear; under these conditions the Kalman filter is globally optimal. For linear measurements/dynamics but without Gaussian distributions, the Kalman filter is not globally optimal, but it still is the linear unbiased minimum variance estimator (i.e. is the optimal out of all linear unbiased estimators) [3]. Gaussian sum filters [4] deal with non-Gaussian distributions. Their additional computational cost and complexity however, make them less attractive than the classical Kalman filter for on-board navigation.

A widely used algorithm in real-time nonlinear estimation is the extended Kalman filter [5] (EKF). The EKF is a nonlinear approximation of the Kalman filter which assumes small estimation errors and approximates them to first order to calculate their covariance matrix. Like the Kalman filter, the EKF is also a linear estimator but relies on the additional assumption that the first order approximation is valid. Algorithms exists that relax both the aforementioned assumptions. Filters with a polynomial update of arbitrary order have been known since the sixties [6], knowledge of moments of the conditional estimation error distribution higher than the second are needed for these updates. Gaussian sum and particle filters have been used for nonlinear problems [7, 8], but they also require the knowledge of the distributions.

Techniques exist to overcome some of the limitations of the EKF linearization assumption. The Gaussian second order filter (GSOF) [9] takes into account second-order terms assuming the error distribution is Gaussian. The iterated extended Kalman filter [5] recursively improves the center of the Taylor series expansion for a better linearization. The unscented Kalman filter [10] (UKF) is

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able to retain higher-order terms of the Taylor series expansion. Underweighting [11] is an *ad hoc* technique to compensate for the second order effects without actually computing them. A recent algorithm, the Recursive Update Filter [12], breaks down the measurement update in many parts, hence avoiding violating linearization assumptions.

The need to address nonlinear effects in space navigation is well known. The Space Shuttle, for example, utilizes an *ad hoc* technique known as underweighting [13, 14]. In the early seventies, in anticipation of Shuttle flights, Lear and others noted that in situations involving large state errors and very precise measurements, the extended Kalman filter's state estimation error covariance decreases more rapidly than the actual state errors. Underweighting was introduced to slow down the convergence of the state estimation error covariance thereby addressing the situation in which the error covariance becomes overly optimistic with respect to the actual state errors. The original work on the application of second-order correction terms led to the determination of the underweighting method by trial-and-error [15].

In this work a spacecraft rendezvous navigation methodology is proposed to address the nonlinearity issue. Rather than slowing down convergence as done in the Space Shuttle, the proposed method utilizes the recursive update filter algorithm to obtain a nonlinear estimate.

RECURSIVE UPDATE FILTER

This section reviews the recursive update filter as introduced in [12].

The Kalman filter's linear update is applied all at once when a measurement becomes available. The idea behind the proposed scheme is to apply the update gradually, allowing to recalculate the Jacobian and to "follow" the nonlinearity of the measurement as the update is applied.

Lower case letters indicate realizations of the random variables. For example $\mathbf{x}$ is the true state which amounts to the value of of the random variable $\mathbf{X}$ for the event under consideration.

Recall the Kalman filter with correlated measurement and process noise [16], define the cross-covariance at time t_k between the true (unknown) state $\mathbf{x}_k$ and the zero-mean measurement noise $\boldsymbol{\eta}_k$ as $\mathbf{C}_k$

$$\mathbf{C}_k = \mathrm{E}\left\{(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\boldsymbol{\eta}_k^{\mathrm{T}}\right\}, \tag{1}$$

where $\hat{\mathbf{x}}_k^-$ is the *a priori* estimated state. To derive the equations of this scheme a linear measurement $\mathbf{y}_k$ is first assumed

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \boldsymbol{\eta}_k. \tag{2}$$

Choosing a linear unbiased update

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-). \tag{3}$$

The optimal gain in terms of minimum variance estimation is given by

$$\mathbf{K}_k = (\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k)\left(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{H}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}\right)^{-1}, \tag{4}$$

where $\mathbf{P}_k^-$ is the *a priori* estimation error covariance matrix and $\mathbf{R}_k$ is the measurement error covariance matrix. It is assumed throughout this work that all errors and noises are zero mean. The updated estimation error covariance is given by

$$\begin{aligned}\mathbf{P}_k^+ =& (\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^-(\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^{\mathrm{T}} - \\ & (\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)\mathbf{C}_k\mathbf{K}_k^{\mathrm{T}} - \mathbf{K}_k\mathbf{C}_k^{\mathrm{T}}(\mathbf{I}_{n\times n} - \mathbf{K}_k\mathbf{H}_k)^{\mathrm{T}},\end{aligned} \tag{5}$$

where $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix, n being the size of the state vector.

Assume the same measurement is processed twice, after the first update the *a posteriori* estimation error is

$$\mathbf{e}_k^{(1)} = \mathbf{x}_k - \hat{\mathbf{x}}_k^{(1)} = (\mathbf{I}_{n\times n} - \mathbf{K}_k^{(1)}\mathbf{H}_k)\mathbf{e}_k^{-} - \mathbf{K}_k^{(1)}\boldsymbol{\eta}_k, \tag{6}$$

where $\mathbf{e}_k^{-} = \mathbf{x}_k - \hat{\mathbf{x}}_k^{-}$ is the *a priori* estimation error. The first optimal gain is

$$\mathbf{K}_k^{(1)} = \mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}. \tag{7}$$

The cross-covariance of $\mathbf{e}_k^{(1)}$ and $\boldsymbol{\eta}_k$ is given by

$$\mathbf{C}_k^{(1)} = -\mathbf{K}_k\mathbf{R}_k. \tag{8}$$

The updated covariance is obtained simplifying Eq. (5) to obtain

$$\mathbf{P}_k^{(1)} = (\mathbf{I}_{n\times n} - \mathbf{K}_k^{(1)}\mathbf{H}_k)\mathbf{P}_k^{-} - \mathbf{K}_k\mathbf{C}_k^{\mathrm{T}}. \tag{9}$$

The cross-covariance between the *a priori* state and the measurement error is denotes as $\mathbf{C}_k^{(0)}$ and is assumed to be zero, i.e. the quantities are uncorrelated and $\mathbf{C}_k^{(0)} = \mathbf{C}_k = \mathbf{O}$. Eq. (9) is only valid when the gain is chosen as the optimal gain, which is not always true in the remaining of this section. Eq. (5) is valid for any choice of $\mathbf{K}_k$.

Processing the same measurement again the second optimal gain is obtained by substituting Eq. (8) into Eq. (4) and replacing $\mathbf{P}_k^{-}$ with $\mathbf{P}_k^{(1)}$

$$\begin{aligned}\mathbf{K}_k^{(2)} = (&\mathbf{P}_k^{(1)}\mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k^{(1)}) \\ &(\mathbf{H}_k\mathbf{P}_k^{(1)} - \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{H}_k\mathbf{C}_k^{(1)} + \mathbf{C}_k^{(1)\mathrm{T}}\mathbf{H}_k^{\mathrm{T}})^{-1},\end{aligned} \tag{10}$$

the resulting optimal gain $\mathbf{K}_k^{(2)}$ is zero since

$$\begin{aligned}\mathbf{P}_k^{(1)}\mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k^{(1)} &= (\mathbf{I}_{n\times n} - \mathbf{K}_k^{(1)}\mathbf{H}_k)\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} - \mathbf{K}_k^{(1)}\mathbf{R}_k \\ &= \mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} - \mathbf{K}_k^{(1)}(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k) \\ &= \mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} - \mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} = \mathbf{O}.\end{aligned} \tag{11}$$

This result is to be expected; after the first update all the information from the measurement is extracted, therefore processing the same measurement again, no additional update should occur.

Assume however that the first update is not optimal, only a fraction of the optimal update is applied

$$\mathbf{K}_k^{(1)} = 0.5\,\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}, \tag{12}$$

with this choice of $\mathbf{K}_k^{(1)}$ the resulting $\mathbf{K}_k^{(2)}$ is not zero. After the first iteration only half of the optimal update is applied. During the second iteration the full $\mathbf{K}_k^{(2)}$ is applied such that the net result after both updates is identical to the standard Kalman algorithm. If three iterations are performed, and each iteration updates one third of the total, the first coefficient is $1/3$. The second coefficient is $1/2$ because the remaining optimal update is two thirds of the total, the last coefficient is 1. This procedure can be expanded to an arbitrary number N of iterations.

For the linear measurement case this algorithm is equivalent to the Kalman filter, making the iterations redundant. The benefits of this approach are evident however for nonlinear measurements. Given a measurement which is a nonlinear function of the state

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \boldsymbol{\eta}_k \tag{13}$$

the algorithm for the nonlinear recursive update is given by Table 1.

Table 1. Recursive update, nonlinear measurements

$\mathbf{C}_k^{(0)} = \mathbf{O}, \;\; \mathbf{P}_k^{(0)} = \mathbf{P}_k^-, \;\; \hat{\mathbf{x}}_k^{(0)} = \hat{\mathbf{x}}_k^-$
for $i = 1$ to N
$\mathbf{H}_k^{(i)} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\hat{\mathbf{x}}_k^{(i-1)}}$
$\mathbf{W}_k^{(i)} = \mathbf{H}_k^{(i)}\mathbf{P}_k^{(i-1)}\mathbf{H}_k^{(i)\mathrm{T}} + \mathbf{R}_k + \mathbf{H}_k^{(i)}\mathbf{C}_k^{(i-1)} + \mathbf{C}_k^{(i-1)\mathrm{T}}\mathbf{H}_k^{(i)\mathrm{T}}$
$\mathbf{K}_k^{(i)} = \gamma_k^{(i)}\big(\mathbf{P}_k^{(i-1)}\mathbf{H}_k^{(i)\mathrm{T}} + \mathbf{C}_k^{(i-1)}\big)\big(\mathbf{W}_k^{(i)}\big)^{-1}$
$\hat{\mathbf{x}}_k^{(i)} = \hat{\mathbf{x}}_k^{(i-1)} + \mathbf{K}_k^{(i)}\big(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k^{(i-1)})\big)$
$\mathbf{P}_k^{(i)} = \big(\mathbf{I}_{n\times n} - \mathbf{K}_k^{(i)}\mathbf{H}_k^{(i)}\big)\mathbf{P}_k^{(i-1)}\big(\mathbf{I}_{n\times n} - \mathbf{K}_k^{(i)}\mathbf{H}_k^{(i)}\big)^{\mathrm{T}} + \mathbf{K}_k^{(i)}\mathbf{R}_k\mathbf{K}_k^{(i)\mathrm{T}} - \big(\mathbf{I}_{n\times n} - \mathbf{K}_k^{(i)}\mathbf{H}_k^{(i)}\big)\mathbf{C}_k^{(i)}\mathbf{K}_k^{(i)\mathrm{T}} - \mathbf{K}_k^{(i)}\mathbf{C}_k^{(i)\mathrm{T}}\big(\mathbf{I}_{n\times n} - \mathbf{K}_k^{(i)}\mathbf{H}_k^{(i)}\big)^{\mathrm{T}}$
$\mathbf{C}_k^{(i)} = \big(\mathbf{I}_{n\times n} - \mathbf{K}_k^{(i)}\mathbf{H}_k^{(i)}\big)\mathbf{C}_k^{(i-1)} - \mathbf{K}_k^{(i)}\mathbf{R}_k$
$\gamma_k^{(i)} = 1/(N+1-i)$
end for
$\mathbf{P}_k^+ = \mathbf{P}_k^{(N)}, \;\; \hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^{(N)}$

As the number of recursions steps N tends to infinity, it is possible to obtain differential equations that govern an update law that continuously re-linearizes. Define the independent real variable $\tau \in [0,\ 1]$. Assume the domain of τ is divided into N intervals, each of length $1/N$. Then the algorithm of Table 1 can be rewritten as

$\mathbf{C}_k(0) = \mathbf{O}, \;\; \mathbf{K}_k(0) = \mathbf{O}, \;\; \mathbf{P}_k(0) = \mathbf{P}_k^-, \;\; \hat{\mathbf{x}}_k(0) = \hat{\mathbf{x}}_k^-$
for $i = 1$ to N
$\tau_i = i/N = \tau_{i-1} + 1/N, \;\; \tau_0 = 0$
$\mathbf{H}_k(\tau_i) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\hat{\mathbf{x}}_k(\tau_{i-1})}$
$\mathbf{C}_k(\tau_i) = \big(\mathbf{I}_{n\times n} - \mathbf{K}_k(\tau_{i-1})\mathbf{H}_k(\tau_i)\big)\mathbf{C}_k(\tau_{i-1}) - \mathbf{K}_k(\tau_{i-1})\mathbf{R}_k$
$\gamma_k(\tau_i) = (1/N)/(1 + 1/N - \tau_i)$
$\mathbf{W}_k(\tau_i) = \mathbf{H}_k(\tau_i)\mathbf{P}_k(\tau_{i-1})\mathbf{H}_k^{\mathrm{T}}(\tau_i) + \mathbf{R}_k + \mathbf{H}_k(\tau_i)\mathbf{C}_k(\tau_i) + \mathbf{C}_k^{\mathrm{T}}(\tau_i)\mathbf{H}_k^{\mathrm{T}}(\tau_i)$
$\mathbf{K}_k(\tau_i) = \gamma_k(\tau_i)\big(\mathbf{P}_k(\tau_{i-1})\mathbf{H}_k^{\mathrm{T}}(\tau_i) + \mathbf{C}_k(\tau_i)\big)\,\mathbf{W}_k^{-1}(\tau_i)$
$\hat{\mathbf{x}}_k(\tau_i) = \hat{\mathbf{x}}_k(\tau_{i-1}) + \mathbf{K}_k(\tau_i)\big[\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k(\tau_{i-1}))\big]$
$\mathbf{P}_k(\tau_i) = \big(\mathbf{I}_{n\times n} - \mathbf{K}_k(\tau_i)\mathbf{H}_k(\tau_i)\big)\mathbf{P}_k(\tau_{i-1})\big(\mathbf{I}_{n\times n} - \mathbf{K}_k(\tau_i)\mathbf{H}_k(\tau_i)\big)^{\mathrm{T}} + \mathbf{K}_k(\tau_i)\mathbf{R}_k\mathbf{K}_k^{\mathrm{T}}(\tau_i) - \big(\mathbf{I}_{n\times n} - \mathbf{K}_k(\tau_i)\mathbf{H}_k(\tau_i)\big)\mathbf{C}_k(\tau_i)\mathbf{K}_k^{\mathrm{T}}(\tau_i) - \mathbf{K}_k(\tau_i)\mathbf{C}_k^{\mathrm{T}}(\tau_i)\big(\mathbf{I}_{n\times n} - \mathbf{K}_k(\tau_i)\mathbf{H}_k(\tau_i)\big)^{\mathrm{T}}$
end for
$\mathbf{P}_k^+ = \mathbf{P}_k(1), \;\; \hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k(1)$

Notice that as $N \to \infty$, $\gamma_k \to 0$, therefore $\mathbf{K}_k \to \mathbf{O}$. A new gain $\mathbf{K}_k^*$ is defined as

$$\begin{aligned}\mathbf{K}_k^*(\tau_i) &= \lim_{N\to\infty} \frac{\mathbf{K}_k(\tau_i)}{\Delta\tau} = \lim_{N\to\infty} \frac{\mathbf{K}_k(\tau_i)}{1/N} \\ &= \frac{1}{1-\tau_i}\left(\mathbf{P}_k(\tau_{i-1})\mathbf{H}_k^{\mathrm{T}}(\tau_i) + \mathbf{C}_k(\tau_i)\right)\mathbf{W}_k^{-1}(\tau_i). \qquad (14)\end{aligned}$$

The change in cross-covariance between time steps is given by

$$\begin{aligned}\Delta\mathbf{C}_k(\tau_i) &= \mathbf{C}_k(\tau_i) - \mathbf{C}_k(\tau_{i-1}) \\ &= -\mathbf{K}_k(\tau_{i-1})\mathbf{H}_k(\tau_i)\mathbf{C}_k(\tau_{i-1}) - \mathbf{K}_k(\tau_{i-1})\mathbf{R}_k. \qquad (15)\end{aligned}$$

Similarly the changes in covariance and state estimate are

$$\Delta\hat{\mathbf{x}}_k(\tau_i) = \mathbf{K}_k(\tau_i)[\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k(\tau_{i-1}))] \qquad (16)$$

$$\begin{aligned}\Delta\mathbf{P}_k(\tau_i) = &-\mathbf{K}_k(\tau_i)\mathbf{H}_k(\tau_i)\mathbf{P}_k(\tau_i) - \mathbf{P}_k(\tau_i)\mathbf{H}_k^{\mathrm{T}}(\tau_i)\mathbf{K}_k^{\mathrm{T}}(\tau_i) \\ &+ \mathbf{K}_k(\tau_i)\mathbf{H}_k(\tau_i)\mathbf{P}_k(\tau_i)\mathbf{H}_k^{\mathrm{T}}(\tau_i)\mathbf{K}_k(\tau_i) \\ &- \mathbf{K}_k(\tau_i)\mathbf{C}_k^{\mathrm{T}}(\tau_i)(\mathbf{I}_{n\times n} - \mathbf{K}_k(\tau_i)\mathbf{H}_k^{\mathrm{T}}(\tau_i) \\ &- (\mathbf{I}_{n\times n} - \mathbf{K}_k(\tau_i)\mathbf{H}_k(\tau_i))\mathbf{C}_k(\tau_i)\mathbf{K}_k^{\mathrm{T}}(\tau_i) \\ &+ \mathbf{K}_k^{\mathrm{T}}(\tau_i)\mathbf{R}_k\mathbf{K}_k^{\mathrm{T}}(\tau_i) \qquad (17)\end{aligned}$$

The evolution of the cross-covariance is given by

$$\begin{aligned}\mathbf{C}_k'(\tau) &= \lim_{N\to\infty} \frac{\mathbf{C}_k(\tau)}{\Delta\tau} \\ &= -\mathbf{K}_k^*(\tau)\mathbf{H}_k(\tau_i)\mathbf{C}_k(\tau) - \mathbf{K}_k^*(\tau)\mathbf{R}_k \qquad (18)\end{aligned}$$

Notice that $\mathbf{K}_k^*(1) = \mathbf{O}$ because when $\tau = 1$ all the information from the measurements has been extracted. The evolutions in covariance and state estimate are

$$\hat{\mathbf{x}}_k'(\tau) = \lim_{N\to\infty} \frac{\hat{\mathbf{x}}_k(\tau)}{\Delta\tau} = \mathbf{K}_k^*(\tau)(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k(\tau))] \qquad (19)$$

$$\begin{aligned}\mathbf{P}_k'(\tau) = &-\mathbf{K}_k^*(\tau)\left(\mathbf{P}_k(\tau)\mathbf{H}_k^{\mathrm{T}}(\tau_i) + \mathbf{C}_k(\tau)\right) - \\ &\left(\mathbf{P}_k(\tau)\mathbf{H}_k^{\mathrm{T}}(\tau_i) + \mathbf{C}_k(\tau)\right)^{\mathrm{T}}\mathbf{K}_k^{*\mathrm{T}}(\tau). \qquad (20)\end{aligned}$$

For the linear measurement case, integrating Eqs. (18), (19), and (20) from $\tau = 0$ to $\tau = 1$ is equivalent to the standard Kalman update. The differential update algorithm in the presence of nonlinear measurements is given by Table 2.

In the nonlinear measurement case the total update is different from the EKF and is generally nonlinear. In the differential formulation of the algorithm the Jacobian is computed continuously, therefore the algorithm "follows" the nonlinearity of the measurement. The EKF assumes a linear update and truncates the Taylor series expansion of the residual to first order, this second assumption can be inadequate in certain situations. The algorithms in Table 1 and Table 2 have either an infinitesimal update or an update arbitrarily small, therefore the linearization assumption can always be made valid. The proposed algorithms are not globally optimal, to solve the nonlinear minimum variance problem the distribution of the errors needs to be defined.

Table 2. Differential update, nonlinear measurements

$\mathbf{C}_k(0) = \mathbf{O}, \;\; \mathbf{P}_k(0) = \mathbf{P}_k^-, \;\; \hat{\mathbf{x}}_k(0) = \hat{\mathbf{x}}_k^-$
Integrate $\mathbf{C}_k$, $\hat{\mathbf{x}}_k$, and $\mathbf{P}_k$ for $\tau = 0$ to 1
$\mathbf{H}_k(\tau) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\hat{\mathbf{x}}_k(\tau)}$
$\mathbf{W}_k(\tau) = \mathbf{H}_k(\tau)\mathbf{P}_k(\tau)\mathbf{H}_k^{\mathrm{T}}(\tau) + \mathbf{R}_k + \mathbf{H}_k(\tau)\mathbf{C}_k(\tau) + \mathbf{C}_k^{\mathrm{T}}(\tau)\mathbf{H}_k^{\mathrm{T}}(\tau)$
$\mathbf{K}_k^*(\tau) = 1/(1-\tau)\big(\mathbf{P}_k(\tau)\mathbf{H}_k^{\mathrm{T}}(\tau) + \mathbf{C}_k(\tau)\big)\,\mathbf{W}_k^{-1}(\tau)$
$\mathbf{C}_k'(\tau) = -\mathbf{K}_k^*(\tau)\big(\mathbf{H}_k(\tau)\mathbf{C}_k(\tau) + \mathbf{R}_k\big)$
$\hat{\mathbf{x}}_k'(\tau) = \mathbf{K}_k^*(\tau)\big[\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k(\tau))\big]$
$\mathbf{P}_k'(\tau) = -\mathbf{K}_k^*(\tau)\big(\mathbf{P}_k(\tau)\mathbf{H}_k^{\mathrm{T}}(\tau) + \mathbf{C}_k(\tau)\big) - \big(\mathbf{P}_k(\tau)\mathbf{H}_k^{\mathrm{T}}(\tau) + \mathbf{C}_k(\tau)\big)^{\mathrm{T}}\mathbf{K}_k^{*\mathrm{T}}(\tau)$
end Integrate
$\mathbf{P}_k^+ = \mathbf{P}_k(1), \;\; \hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k(1).$

In practical implementations the differential update needs to be computed numerically which requires an integration step and inevitably introduces numerical errors. In the author's experience, it is more accurate to use 4 recursions from Table 1 than a single step fourth order Runge-Kutta propagation from Table 2; the two approaches require the same number of function evaluations. Using N recursions is equivalent to solving the differential update with N Euler steps. Since $\mathbf{K}_k(1) = \mathbf{O}$ numerical integration schemes that do not evaluate the derivative at the end of the step seem to be more precise.

The number of iterations is a user-defined parameter that needs to be selected by addressing two conflicting design objectives. On the one hand the number of iterations should be chosen high to improve performance; the more iterations the more often the algorithm re-linearizes and the better the non-linearity of the measurement is followed. On the other hand whenever computational time is of concern it is desirable to reduce the number of iterations. In general, the higher the degree of nonlinearity of the measurement, the more iterations are needed. A good indicator of the performance of the algorithm is the post-update residual. The residual (actual measurement minus estimated measurement) where the estimated measurement is computed with the updated state, should match its predicted covariance. Discrepancies between the two indicate the nonlinear effects are of concern and more iterations are needed.

NUMERICAL RESULTS

This section contains a spacecraft rendezvous example. The relative trajectory in local horizontal local vertical (LVLH) coordinates is shown in Figure 1. Z is radial direction pointing down, X is downrange pointing in the same direction as velocity, and Y is the out-of-plane direction to complete a right hand triad. Fig. 1, shows the relative trajectory, which is a V-bar approach (i.e. starting along the target's velocity direction) from 100 meters in front of the target. The estimated state $\mathbf{x}$ is relative position and velocity between chaser and target in the LVLH frame, the initial filter covariance $\mathbf{P}_0$ is obtained setting the state uncertainty to 10 m in position and 0.05 m/s in velocity per axis.

$$\mathbf{P}_0 = \begin{bmatrix} 10^2\mathbf{I}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & 0.05^2\mathbf{I}_{3\times3} \end{bmatrix} \tag{21}$$

The dynamics are governed by the CW equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \boldsymbol{\nu} \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{I}_{3\times3} \\ \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} \tag{23}$$

and

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -n^2 & 0 \\ 0 & 0 & 3n^2 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 2n \\ 0 & 0 & 0 \\ -2n & 0 & 0 \end{bmatrix},$$

with $n = 0.0011$rad/s, the orbital angular rate of the target in a circular orbit. The process noise is given by

$$\mathrm{E}\{\boldsymbol{\nu}\} = \mathbf{0} \qquad \mathrm{E}\left\{\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}\right\} = \begin{bmatrix} \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & 10^{-9}\mathbf{I}_{3\times3} \end{bmatrix} \tag{24}$$

A LIDAR is the primary relative navigation sensor providing range to the target and two bearing angles. The range measurement ρ has 0.1 m accuracy and the bearing angles have 0.1 deg accuracy. The bearing angles are azimuth $\alpha = \tan^{-1}(\mathbf{x}(1)/\mathbf{x}(2))$ and elevation $\epsilon = \sin^{-1}(\mathbf{x}(3)/\rho)$.

$$\mathbf{y} = \begin{bmatrix} \rho \\ \alpha \\ \epsilon \end{bmatrix} + \boldsymbol{\eta} \tag{25}$$

$$\mathrm{E}\{\boldsymbol{\eta}\} = \mathbf{0}, \qquad \mathrm{E}\left\{\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}}\right\} = \begin{bmatrix} 0.1^2 & \mathbf{O}_{1\times2} \\ \mathbf{O}_{2\times1} & (0.1\pi/180)^2\mathbf{I}_{2\times2} \end{bmatrix} \tag{26}$$

One hundred runs are performed in which the initial estimation error and the sensors errors are dispersed using zero mean Gaussian independent random variables. Fig. 2 shows the performance of the EKF, the lighter lines are the 100 samples of the estimation error, while the thicker black lines are the EKF predicted standard deviations. The plot shows that the EKF is overly optimistic in predicting its performance. Eventually however the EKF is able to recover. Under similar but more severe circumstances, it is also possible that the EKF would diverge all together [11].

Fig. 3 shows the performance of 10 recursive updates using the proposed algorithm. It can be seen that the filter outperforms the EKF during the first few hundred seconds. The recursive update filter is also able to predict its performance as most error samples are within the 1σ prediction and all of them are within the 3σ values. The performance of the two algorithms in estimating velocity is very similar to that of the position shown in Figs. 2 and 3.

CONCLUSIONS

In this paper the recursive update filter is reviewed. It is well known that very accurate measurements, such as LIDARs during a spacecraft rendezvous, can cause inconsistency in a EKF estimate or even divergence. This is fact is due to the nonlinear nature of measurements and the linearized approximation of the EKF. Currently many spacecrafts implement “underwighting” a technique to reduce the measurement update in order to prevent divergence and gradually converge. In contrast to the EKF, the recursive update filter is nonlinear in nature, and can “follow” the nonlinearity of the measurement. This property allows the recursive update filter to fully extract all the information from the measurement and to avoid inconsistency and divergence.

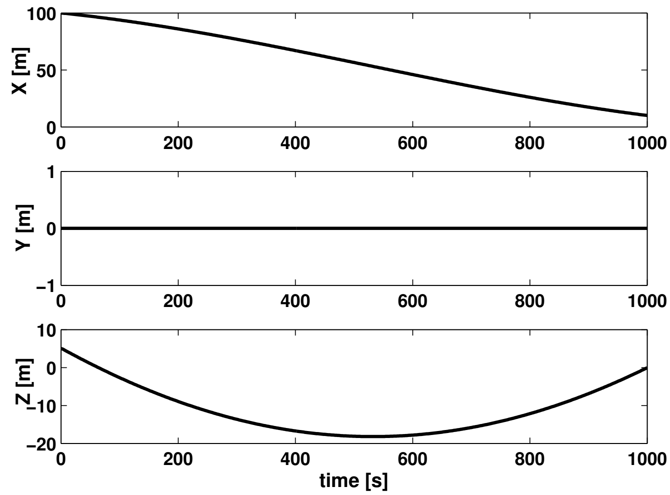

Figure 1. Trajectory.

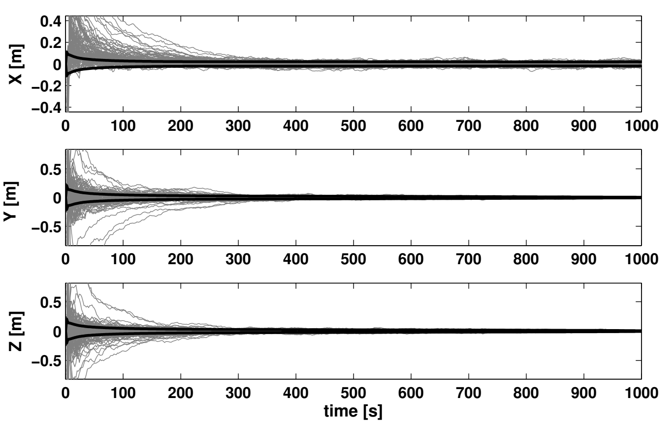

Figure 2. EKF Estimation Error and Predicted Covariance.

A numerical example showing a spacecraft rendezvous scenario is shown. While the EKF provides inconsistent estimate, the numerical results suggest that the recursive update filter is a viable method to design a rendezvous navigation filter in the presence of LIDAR measurements.

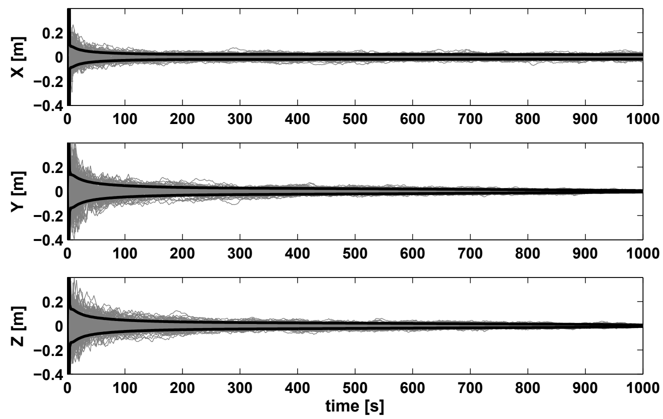

Figure 3. Recursive Update Filter Estimation Error and Predicted Covariance.

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