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THE INITIAL ORBIT DETERMINATION (IOD) PROBLEM WITH RANGE, RANGE-RATE AND ANGLES

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An analytical formulation of the Initial Orbit Determination (IOD) problem for range, range-rate and angle measurements. This approach involves the use of the Lagrange interpolating polynomials to obtain the position and velocity estimate. In addition the covariance matrix associated with the state estimate is provided.

INTRODUCTION

The Initial Orbit Determination (IOD) problem has been tacked since Laplace, Euler and Gauss in the service of determining the orbits of planets, asteroids, and comets. The measurements that established the orbit characteristics were angles over long arcs.[1] The Space Age brought two new measurement types to bear: range and range-rate.[2,3] Initially these measurements were provided by radars through skin-tracking. Soon, however, transponders allowed better measurement accuracy and availability. To date, there are three types of IOD problems: angles only, range and angles, and range and range-rate.

The Orion program exposed the need for an additional capability in the case of an extended loss of communication: when the measurements included antenna angles (azimuth and elevation or right-ascension and declination), range and range-rate. This paper contains the details of this new methodology. We will begin with state determination with range, range-rate and angle measurements.

First the state estimate, given a sequence of these four time-synchronized measurements, will be analytically developed. Next the covariance matrix associated with this state estimate will be analytically developed. Finally, these will be verified by Monte Carlo simulations.

THE STATE AND COVARIANCE DETERMINATION FOR RANGE, RANGE-RATE AND STATION ANGLE MEASUREMENTS

We assume that we will have (at least) three sets of (range, range-rate and station angles (azimuth and elevation) measurements at times t_i. We further assume that over a short arc, the vehicle is moving in Keplerian motion.

Given a set of measurement epochs, $t_1, t_2, \cdots, t_n$, we seek to find the state vector at t_i within the interval ($t_i > t_i \,\mathrm{and}\, t_i < t_n$), given a set of Line of Sight (LOS) vectors ($\mathbf{u}_{LOS}$) from the ground station (with position vector $\mathbf{r}_{GS}$) to the vehicle ($\mathbf{r}$) as

$$\rho\, \mathbf{u}^I_{LOS} \quad = \quad \mathbf{r}^I - \mathbf{r}^I_{GS} \tag{1}$$

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where ρ is the range to the vehicle (from the ground station) to get

$$\mathbf{r}^I = \rho\,\mathbf{u}^I_{LOS} + \mathbf{r}^I_{GS} \tag{2}$$

Differentiating the above equation we get

$$\dot{\mathbf{r}}^I = \dot{\rho}\,\mathbf{u}^I_{LOS} + \rho\,\dot{\mathbf{u}}^I_{LOS} + \dot{\mathbf{r}}^I_{GS} \tag{3}$$

The unknowns here are $\dot{\mathbf{u}}^I_{LOS}$ and $\dot{\mathbf{r}}^I$. Notice too, we haven't made any assumptions as to the type of motion (Keplerian or non-Keplerian).

We introduce the general concept of interpolation introduced by Lagrange in support of celestial mechanics. Linear interpolation is a special case of the Lagrange interpolating polynomials. The Lagrange interpolating formula, given a series of points $\boldsymbol{\xi}_i$ at times t_i, is

$$\boldsymbol{\xi}(t) = \sum_{i=1}^{n} \boldsymbol{\xi}_i \frac{\prod_{k\neq i}(t-t_k)}{\prod_{k\neq i}(t_i-t_k)} \tag{4}$$

so that for three unit line-of-sight position vectors, $\mathbf{u}_{LOS_1}$, $\mathbf{u}_{LOS_2}$, and $\mathbf{u}_{LOS_3}$we get

$$\mathbf{u}^I_{LOS}(t) = \mathbf{u}^I_{LOS_1}\frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + \mathbf{u}^I_{LOS_2}\frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + \mathbf{u}^I_{LOS_3}\frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)} \tag{5}$$

so that $\dot{\mathbf{u}}^I_{LOS_i}(t)$ and $\ddot{\mathbf{u}}_{LOS_i}(t)^I$ are

$$\dot{\mathbf{u}}^I_{LOS}(t) = \mathbf{u}^I_{LOS_1}\frac{(2t-t_2-t_3)}{(t_1-t_2)(t_1-t_3)} + \mathbf{u}^I_{LOS_2}\frac{(2t-t_1-t_3)}{(t_2-t_1)(t_2-t_3)} + \mathbf{u}^I_{LOS_3}\frac{(2t-t_1-t_2)}{(t_3-t_1)(t_3-t_2)} \tag{6}$$

$$\ddot{\mathbf{u}}^I_{LOS}(t) = \mathbf{u}^I_{LOS_1}\frac{2}{(t_1-t_2)(t_1-t_3)} + \mathbf{u}^I_{LOS_2}\frac{2}{(t_2-t_1)(t_2-t_3)} + \mathbf{u}^I_{LOS_3}\frac{2}{(t_3-t_1)(t_3-t_2)} \tag{7}$$

Of course we are not limited to just three unit vectors but for illustrative purposes we used three in the prior description to make the method a bit more tractable. Thus, if n points are used for interpolation, $\dot{\mathbf{u}}^I_{LOS_i}(t)$ is

$$\dot{\mathbf{u}}^I_{LOS}(t) = \sum_{i=1}^{n} \mathbf{u}^I_{LOS_i}\dot{\mathcal{L}}_i(t) \tag{8}$$

The Lagrange Interpolating Polynomials and Their Derivatives

Recall the general form for interpolating with n^{th}-order Lagrange interpolating polynomials as

$$\boldsymbol{\xi}(t) = \sum_{j=1}^{n} \boldsymbol{\xi}_j \frac{\prod_{k=1,k\neq j}^{n}(t-t_k)}{\prod_{k=1,k\neq j}^{n}(t_j-t_k)} = \sum_{j=1}^{n} \boldsymbol{\xi}_j \mathcal{L}_j(t) \tag{9}$$

where the Lagrange interpolating polynomials are

$$\mathcal{L}_j(t) \triangleq \frac{\prod_{k=1,k\neq j}^{n}(t-t_k)}{\prod_{k=1,k\neq i}^{n}(t_j-t_k)} \tag{10}$$

The (time) derivative of the Lagrange interpolating polynomials is

$$\dot{\mathcal{L}}_j(t) \;=\; \mathcal{L}_j(t)\left(\sum_{k=1,k\neq j}^{n} \frac{1}{t-t_k}\right) \tag{11}$$

A note about using Lagrange interpolating polynomials. For high order polynomials, the Lagrange interpolating polynomials suffers from the Runge phenomenon so it is best to pick the interpolating time in the middle of the data arc.

As well the Lagrange interpolating polynomial is best used for a single ground station. The same process can be performed when another ground station comes into view and the two solutions can be combined statistically.

The Use of Lagrange Interpolating Polynomials in Initial Orbit Determination

But we need to be a bit careful because the unit vectors are computed in the station-fixed frame of station j according to

$$\mathbf{u}^I_{LOS_j}(t_i) = \mathbf{T}^I_{GS_j}(t_i)\mathbf{u}^{GS_j}_{LOS_i} \tag{12}$$

where the azimuth and elevation of the ground station antenna are α_i and δ_i as

$$\mathbf{u}^{GS_j}_{LOS_i} \;\triangleq\; \begin{bmatrix} \cos\alpha_i \cos\delta_i \\ \sin\alpha_i \cos\delta_i \\ \sin\delta_i \end{bmatrix} \tag{13}$$

Likewise the position vectors of the ground stations are expressed as

$$\mathbf{r}^I_{GS}(t) = \mathbf{T}^I_{ECEF}\mathbf{r}^{ECEF}_{GS} \tag{14}$$

where the vector $\mathbf{r}^{ECEF}_{GS}$ is the Earth fixed position of the ground station (and is expected to be constant, except for the location error) and $\mathbf{T}^I_{ECEF}$ is the transformation matrix from planet-fixed (ECEF) to inertial, so that

$$\dot{\mathbf{r}}^I_{GS}(t) = \boldsymbol{\omega}^I_E \times \mathbf{T}^I_{ECEF}\mathbf{r}^{ECEF}_{GS} \tag{15}$$

where $\boldsymbol{\omega}^I_E$ is the rotation vector of the Earth.

The State Determination

So now we have everything we need. If we want to get the position and velocity of the vehicle at time t_i where $t_i \in (t_1, t_n)$ and we have $\rho(t_i)$, $\dot{\rho}(t_i)$, $\alpha(t_i)$, and $\delta(t_i)$ measurements from ground station j at time t_i, we get the following

$$\mathbf{r}^I(t_i) \;=\; \rho(t_i)\mathbf{T}^I_{GS_j}(t_i)\begin{bmatrix} \cos\alpha(t_i)\cos\delta(t_i) \\ \sin\alpha(t_i)\cos\delta(t_i) \\ \sin\delta(t_i) \end{bmatrix} + \mathbf{T}^I_{ECEF}(t_i)\mathbf{r}^{ECEF}_{GS_j} \tag{16}$$

$$\mathbf{v}^I(t_i) \;=\; \dot{\rho}(t_i)\,\mathbf{T}^I_{GS_j}(t_i)\begin{bmatrix} \cos\alpha(t_i)\cos\delta(t_i) \\ \sin\alpha(t_i)\cos\delta(t_i) \\ \sin\delta(t_i) \end{bmatrix} + \rho(t_i)\,\dot{\mathbf{u}}^I_{LOS}(t_i) + \boldsymbol{\omega}^I_E \times \mathbf{T}^I_{ECEF}(t_i)\mathbf{r}^{ECEF}_{GS_j} \tag{17}$$

The Covariance Determination

In order to determine the covariance, we take variations of Eqs (16) and (17).

First for the position, we establish the following orthonormal LOS system

$$\mathbf{u}_{LOS_x}(t_i) = \begin{bmatrix} \cos\alpha(t_i)\cos\delta(t_i) \\ \sin\alpha(t_i)\cos\delta(t_i) \\ \sin\delta(t_i) \end{bmatrix} \tag{18}$$

$$\mathbf{u}_{LOS_y}(t_i) = \cos\delta(t_i)\begin{bmatrix} -\sin\alpha(t_i) \\ \cos\alpha(t_i) \\ 0 \end{bmatrix} \tag{19}$$

$$\mathbf{u}_{LOS_z}(t_i) = -\begin{bmatrix} \cos\alpha(t_i)\sin\delta(t_i) \\ \sin\alpha(t_i)\sin\delta(t_i) \\ -\cos\delta(t_i) \end{bmatrix} \tag{20}$$

so that the variation of the position vector (as found in Eq. (16)) becomes

$$\begin{aligned} d\mathbf{r}(t_i) &= d\rho(t_i)\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_x}(t_i) + \rho(t_i)\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_y}(t_i)d\alpha(t_i) \\ &\quad +\rho(t_i)\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_z}(t_i)d\delta(t_i) + \mathbf{T}^I_{ECEF}(t_i)d\mathbf{r}^{ECEF}_{GS_j} \end{aligned} \tag{21}$$

so that if $\mathbf{P_{rr}}(t_i) \triangleq E\left[d\mathbf{r}(t_i)d\mathbf{r}^T(t_i)\right]$ and noting the orthogonality of $\mathbf{u}_{LOS_x}(t_i)$, $\mathbf{u}_{LOS_y}(t_i)$ and $\mathbf{u}_{LOS_z}(t_i)$ and the independence of the measurements and the errors in the ground station location, we get

$$\begin{aligned} \mathbf{P_{rr}}(t_i) &= \sigma^2_\rho\,\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_x}(t_i)\mathbf{u}^T_{LOS_x}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i) + \sigma^2_\alpha\rho^2\,\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_y}(t_i)\mathbf{u}^T_{LOS_y}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i) \\ &\quad +\sigma^2_\delta\rho^2\,\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_z}(t_i)\mathbf{u}^T_{LOS_z}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i) + \mathbf{T}^I_{ECEF}(t_i)\mathbf{P}_{\mathbf{r}^{ECEF}_{GS_j}}\mathbf{T}^{I^T}_{ECEF}(t_i) \end{aligned} \tag{22}$$

The variation of the velocity vector is much more involved. It is

$$\begin{aligned} d\mathbf{v}(t_i) &= d\dot\rho(t_i)\,\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_x}(t_i) + \dot\rho(t_i)\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_y}(t_i)d\alpha(t_i) + \dot\rho(t_i)\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_z}(t_i)d\delta(t_i) \\ &\quad + d\rho(t_i)\,\dot{\mathbf{u}}^I_{LOS}(t_i) + \rho(t_i)\,d\dot{\mathbf{u}}^I_{LOS}(t_i) + \boldsymbol{\omega}^I_E \times \mathbf{T}^I_{ECEF}(t_i)d\mathbf{r}^{ECEF}_{GS_j} \end{aligned} \tag{23}$$

We also denote the cross-product in terms of the cross product matrix

$$\mathbf{S}_{\boldsymbol{\omega}^I_E} \triangleq \left[\boldsymbol{\omega}^I_E\times\right] \tag{24}$$

and $\mathbf{S}^T_{\boldsymbol{\omega}^I_E} = -\mathbf{S}_{\boldsymbol{\omega}^I_E}$.

What remains is to obtain the variation of $\dot{\mathbf{u}}^I_{LOS}(t_i)$. We find that

$$\begin{aligned} d\dot{\mathbf{u}}^I_{LOS}(t_i) &= \sum_{k=1}^n \dot{\mathcal{L}}_k(t_i)d\mathbf{u}^I_{LOS_k} = \sum_{k=1}^n \dot{\mathcal{L}}_k(t)\mathbf{T}^I_{GS_j}(t_k)d\mathbf{u}^{GS_j}_{LOS_k} \\ &= \sum_{k=1}^n \dot{\mathcal{L}}_k(t_i)\mathbf{T}^I_{GS_j}(t_k)d\begin{bmatrix} \cos\alpha^j_k\cos\delta^j_k \\ \sin\alpha^j_k\cos\delta^j_k \\ \sin\delta^j_k \end{bmatrix} \\ &= \sum_{k=1}^n \dot{\mathcal{L}}_k(t_i)\mathbf{T}^I_{GS_j}(t_k)\left\{\mathbf{u}^j_{LOS_y}(t_k)d\alpha^j_k + \mathbf{u}^j_{LOS_z}(t_k)d\delta^j_k\right\} \end{aligned} \tag{25}$$

where the notation $\mathbf{u}^j_{LOS_y}(t_k)$ denotes the $y-$LOS unit vector of the j^{th} station at time t_k.

We assume that the angle errors, $d\alpha^j_k$ and $d\delta^j_k$, are uncorrelated with one another and uncorrelated over time.

With this in hand, the velocity covariance is

$$\begin{aligned}\mathbf{P_{vv}}(t_i) \;=\; & \sigma^2_{\dot\rho}\,\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_x}(t_i)\mathbf{u}^T_{LOS_x}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i) + \dot\rho^2(t_i)\sigma^2_\alpha\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_y}(t_i)\mathbf{u}^T_{LOS_y}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i)\\ & +\dot\rho^2(t_i)\sigma^2_\delta\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_z}(t_i)\mathbf{u}^T_{LOS_z}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i) + \sigma^2_\rho\dot{\mathbf{u}}^I_{LOS}(t_i)\dot{\mathbf{u}}^{I^T}_{LOS}(t_i)\\ & +\rho^2(t_i)\sum_{k=1}^{n}\dot{\mathcal{L}}^2_k(t_i)\mathbf{T}^I_{GS_j}(t_k)\Big[\sigma^2_\alpha\mathbf{u}^j_{LOS_y}(t_k)\mathbf{u}^{j^T}_{LOS_y}(t_k) + \sigma^2_\delta\mathbf{u}^j_{LOS_z}(t_k)\mathbf{u}^{j^T}_{LOS_z}(t_k)\Big]\mathbf{T}^{I^T}_{GS_j}(t_k)\\ & -\mathbf{S}_{\boldsymbol{\omega}^I_E}\mathbf{T}^I_{ECEF}(t_i)\mathbf{P}_{\mathbf{r}^{ECEF}_{GS_j}}\mathbf{T}^{I^T}_{ECEF}(t_i)\mathbf{S}_{\boldsymbol{\omega}^I_E} \end{aligned} \tag{26}$$

The cross-covariance between position and velocity is

$$\begin{aligned}\mathbf{P_{rv}}(t_i) \;=\; & \sigma^2_\rho\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_x}(t_i)\dot{\mathbf{u}}^{I^T}_{LOS}(t_i) - \mathbf{T}^I_{ECEF}(t_i)\mathbf{P}_{\mathbf{r}^{ECEF}_{GS_j}}\mathbf{T}^{I^T}_{ECEF}(t_i)\mathbf{S}_{\boldsymbol{\omega}^I_E}\\ & +\sigma^2_\alpha\rho(t_i)\left(\dot\rho(t_i) + \rho(t_i)\dot{\mathcal{L}}_k(t_i)\right)\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_y}(t_i)\mathbf{u}^T_{LOS_y}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i)\\ & +\sigma^2_\delta\rho(t_i)\left(\dot\rho(t_i) + \rho(t_i)\dot{\mathcal{L}}_k(t_i)\right)\mathbf{T}^I_{GS_j}(t_i)\mathbf{u}_{LOS_z}(t_i)\mathbf{u}^T_{LOS_z}(t_i)\mathbf{T}^{I^T}_{GS_j}(t_i)\end{aligned} \tag{27}$$

APPLICATION TO THE ARTEMIS 1 CONTINGENCY INITIAL ORBIT DETERMINATION PROBLEM

There are various data types that are used (and produced) by the DSN and are part of the TRK-2-34 DSN Data Archival Format[‡‡]. These are denoted by the prefix 'DT' for *Data Type*.

For Orion, the primary data type is Doppler and range (at isolated points, but on demand in emergencies. Angle (azimuth/elevation or RAI/DEC) is not used in the 'usual' OD processing. Whereas, it is part of the DT-8 message (as part of the TRK-2-34 format), it is available. However, it is not accurate, having a precision of 0.1 degrees. For other antennas which give more precise angle measurements, it will be far more beneficial.

For this instance the following measurement accuracies are used:

$$\begin{aligned}\sigma_\rho &= 10\,\mathrm{m}\\ \sigma_{\dot\rho} &= 0.005\,\mathrm{m/s}\\ \sigma_\alpha &= 0.1^\circ\\ \sigma_\delta &= 0.1^\circ\end{aligned}$$

with $n = 7$ and 10,000 Monte Carlo samples at each time step.

For the first ground navigation pass, with these measurement accuracies, Figure[?] shows the accuracy of the IOD estimate in terms of its standard deviation. The Monte Carlo (sample) standard deviation and the analytical standard deviation as expressed in Eqs.(22) and (26) are plotted together. It is seen that they match. However, with these errors for the DSN angles, for a typical outbound Artemis 1 trajectory, the position and velocity accuracies increase as a function of the increasing distance. A lower angle accuracy would yield much different results.

[‡‡] *TRK-2-34 DSN Tracking System Data Archival Format*, DSN No. 820-013, TRK-2-34, Rev. R, JPL D-76488, June 3, 2021, CL#21-2444.

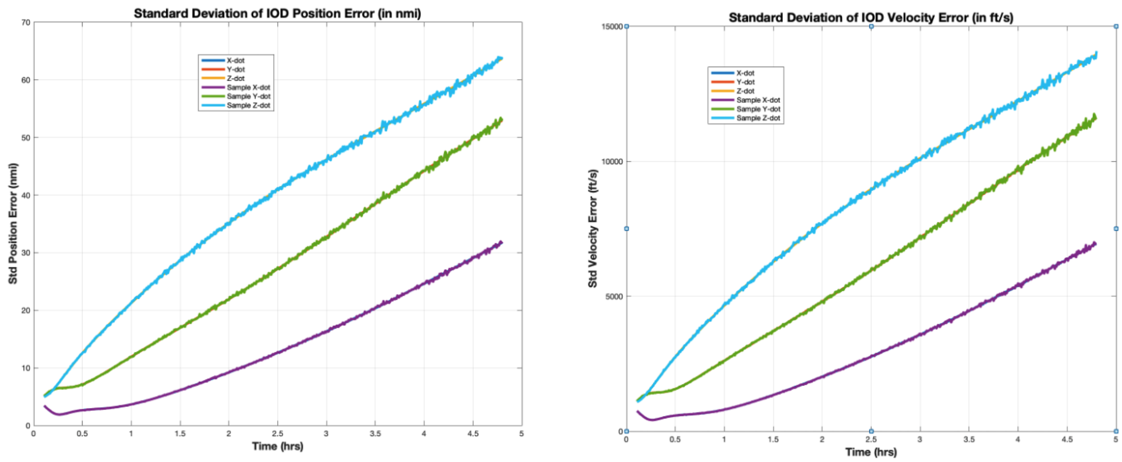

Figure 1. Position and Velocity standard deviation for Earth-Moon trajectory with $n = 7$ for the initial portion of the trajectory

CONCLUSIONS

This paper has presented a novel analytical method for the state solution of the IOD problem with range, range-rate and angle measurements. It also gave details about the analytical formulation of the covariance matrix associated with the state estimate. It has been demonstrated on a trajectory analogous to the outbound leg of the Orion Artemis 1 mission.

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