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ATTITUDE ESTIMATION OF PLANAR ROTATIONS USING REDUNDANT GYROSCOPES

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Accurate heading angle estimation is essential for autonomous robotic navigation, particularly in scenarios where external reference signals such as GPS are unavailable. Traditional approaches rely on a single gyroscope for propagating the heading angle, while redundant gyros are used solely for fault detection and isolation. Although effective for fault tolerance, this approach underutilizes available sensor data, potentially limiting estimation precision. This paper proposes two methods for fully exploiting redundant gyroscope measurements to enhance heading angle estimation in planar environments. The first method uses a single gyroscope for state propagation while incorporating additional gyros in the update step. The second method computes an average angular velocity across all gyroscopes and uses it for propagation. To preserve the observability of individual gyro biases, critical for fault detection, a state transformation is introduced, ensuring that only one bias influences the propagated state while the others appear in the measurement residuals. It is shown that the two methods are mathematically equivalent. Numerical simulations and Monte Carlo analyses confirm that the proposed estimators improve heading accuracy while retaining fault detection capabilities.

INTRODUCTION

Accurate heading angle estimation is a fundamental problem in robotics, with a wide range of applications including mobile robot localization, simultaneous localization and mapping (SLAM), visual odometry, and object pose estimation.[1–3] This problem is especially critical in odometry-based systems, where even a small, temporary orientation error can lead to a continually growing position error. In planar environments, the robot's orientation is typically represented by a single angle $\theta \in [-\pi, \pi)$, corresponding to the rotation around a reference axis.

Rotation estimation in robotics often relies on sensor fusion techniques that combine data from multiple sources. A common approach is odometry-based estimation, where wheel encoders provide coarse orientation updates that are subsequently refined using inertial measurements from gyroscopes.[4] However, these methods are prone to cumulative errors and drift over time, highlighting the need for external references such as visual landmarks or GPS measurements. In the space domain, the standard method for this correction is the use of star tracker measurements. While star trackers are typically used to estimate full 3D orientation, for estimating a planar rotation, like for example in the case of a lunar rover, their measurements can be mapped to directly observe the heading angle.[5]

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In the context of attitude estimation for aerospace applications, sensor redundancy is critical for ensuring reliable fault detection and isolation. The typical approach relies on a single gyroscope measurement for state propagation, using the so-called dynamic model replacement method,[6] while the remaining gyroscopes are reserved exclusively for fault detection and isolation. In particular, when three IMUs are available, it is possible to implement a center-select algorithm.[7,8] This algorithm selects a reference value, typically the median, from the set of redundant sensors. Any measurement that deviates beyond a predefined threshold is classified as faulty and subsequently discarded. While effective, this approach underutilizes the available sensor data, potentially limiting the precision of the estimation. Recent research has demonstrated that averaging measurements from multiple IMUs can yield a more precise estimate.[9,10] In particular, it has been shown that by combining data from six low-grade gyroscopes, it is possible to achieve a level of accuracy comparable to that of a tactical-grade gyro.[9] While this approach enhances measurement precision, it often overlooks the observability of individual gyro biases. Proper estimation of these biases is essential for detecting and isolating faults, as unobservable biases can degrade system performance and allow sensor failures to go undetected.

The goal of this paper is to propose an algorithm for estimating planar rotations by leveraging all available gyroscope measurements. The first contribution is to demonstrate that heading angle propagation can be effectively achieved using an averaged angular velocity measurement. Under certain conditions, this approach is shown to be equivalent to a scheme where one gyroscope is used for state propagation and the others for state updates. The second contribution is an estimation filter that utilizes the average of all gyroscope measurements for state propagation while retains the observability of individual gyro biases. The novelty of the proposed algorithm lies in its measurement update step: to retain bias observability for all gyroscopes, the state is updated using the differences between individual angular velocity measurements. A full analysis of the correlation between the average measurement used for propagation and the differential measurements used for the update is presented, and it is proven that the two methods are mathematically equivalent.

The remainder of this paper is organized as follows. First, the implementation of the two proposed methods is introduced in a scenario with bias-free gyros. The analysis is then extended to incorporate biased angular measurements. Numerical simulations and Monte Carlo analyses are conducted to evaluate the proposed methods. Finally, conclusions are drawn.

PLANAR ROTATION WITH BIAS-FREE GYRO MEASUREMENTS

This section compares different methodologies for estimating a planar rotation using measurements from two bias-free gyros. Specifically, we want to estimate the heading angle $\theta(t)$ whose continuous-time dynamics are

$$\dot{\theta}(t) = \omega(t). \tag{1}$$

Assuming a slowly varying angular velocity and a sufficiently high sampling rate, the discrete-time dynamics can be approximated using a backward Euler step:[11]

$$\theta_k = \theta_{k-1} + \omega_k \Delta t, \tag{2}$$

where $\Delta t = t_k - t_{k-1}$. At time k, measurements from two gyros are available. Throughout this paper, the superscript (i) denotes a quantity associated with the *i-th* gyro and a tilde $\widetilde{(.)}$ is used to denote that $(.)$ is a measured quantity. The measurement model is

$$\widetilde{\omega}_k^{(i)} = \frac{1}{\Delta t}\left(\theta_k - \theta_{k-1}\right) + \nu_k^{(i)}, \quad i = 1, 2. \tag{3}$$

where $\nu_k^{(i)} \sim \mathcal{N}(0, \sigma_{(i)}^2)$ is an i.i.d. gaussian sequence. Furthermore, the two gyros are assumed to be uncorrelated, hence

$$\mathbb{E}[\nu_k^{(i)} \nu_k^{(j)}] = \sigma_{(i)}^2 \delta_{ij}, \tag{4}$$

where the noise standard deviation is associated with the Angular Random Walk (ARW) and the sampling rate through the relation

$$\sigma_{(i)} = \frac{\text{ARW}^{(i)}}{\sqrt{\Delta t}}. \tag{5}$$

The goal is estimating the angle θ_k at time k, given a prior $\theta_{k-1} \sim \mathcal{N}(\hat{\theta}_{k-1}, P_{k-1})$ at time $k-1$ and measurements $\widetilde{\omega}_k^{(1)}$, $\widetilde{\omega}_k^{(2)}$ at time k. Examining Equations (2), (3), it is possible to infer that the angular velocity provided by the gyros can either be used as an input to propagate the state estimate from time $k-1$ to k or as a measurement to update the estimate at time k. This section analyzes this duality and explores the differences, if any, between using the gyro data as an input versus as a measurement.

The dynamics reported in Equation (2) and the measurement function in Equation (3) are linear with respect to the state, and the prior estimate is assumed to follow a Gaussian distribution. For linear Gaussian systems, the Kalman Filter[12] has been proven to be the optimal estimator in both the Maximum a Posteriori (MAP) sense and the Minimum Mean Square Error (MMSE) sense.[13] Therefore, we use the Kalman Filter framework to compare the proposed methodologies.

Time and Measurement Updates Using Distinct Gyroscopes

At time k, we have two angular velocity measurements. Since angular velocity can be used either as an input to the dynamics or as a measurement, a natural approach is to use one angular velocity measurement to propagate the estimate of the angular state and the other to update this estimate. Specifically, we use the measurement $\widetilde{\omega}_k^{(1)}$ from gyro 1 as an input and the measurement $\widetilde{\omega}_k^{(2)}$ from gyro 2 as an observation. The final estimate remains unchanged if the roles of the two gyros are reversed.

A generic linear system, subjected to uncertain input and no process noise, can be described as:

$$\boldsymbol{x}_k = \Phi \boldsymbol{x}_{k-1} + B \boldsymbol{u}_{k-1}. \tag{6}$$

Given an initial gaussian estimate $\boldsymbol{x}_{k-1} \sim \mathcal{N}(\hat{\boldsymbol{x}}_{k-1}, P_{k-1})$, and a gaussian input $\boldsymbol{u}_{k-1} \sim \mathcal{N}(\hat{\boldsymbol{u}}_{k-1}, U)$ the estimate remains gaussian throughout the process, i.e. $\boldsymbol{x}_k \sim \mathcal{N}(\bar{\boldsymbol{x}}_k, \bar{P}_k)$, where:

$$\bar{\boldsymbol{x}}_k = \Phi \hat{\boldsymbol{x}}_{k-1} + B \hat{\boldsymbol{u}}_{k-1}, \tag{7}$$

$$\bar{P}_k = \Phi P_{k-1} \Phi^T + B U B^T. \tag{8}$$

For planar rotation, when the input is the angular velocity measured by gyro 1, we get

$$\hat{\boldsymbol{u}}_{k-1} = \widetilde{\omega}_k^{(1)}, \tag{9}$$

$$\Phi = 1, \tag{10}$$

$$B = \Delta t, \tag{11}$$

$$U = \sigma_{(1)}^2, \tag{12}$$

and, consequently, the propagated estimate at time k is $\theta_k \sim \mathcal{N}(\bar{\theta}_k, \bar{P}_k)$ where:

$$\bar{\theta}_k = \hat{\theta}_{k-1} + \widetilde{\omega}_k^{(1)} \Delta t, \tag{13}$$

$$\bar{P}_k = P_{k-1} + \Delta t^2 \sigma_{(1)}^2. \tag{14}$$

To incorporate the information from gyro 2, it is necessary to modify the Kalman update[12] to account for the fact that the measurement $\boldsymbol{z}_k$ is a linear function of the state at both time k and $k-1$:

$$\boldsymbol{z}_k = H\boldsymbol{x}_k + J\boldsymbol{x}_{k-1} + \boldsymbol{v}_k, \tag{15}$$

where $\boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, R)$. The dependence of the measurement on the state at time $k-1$ introduces a correlation between the error state and the measurement error, which must be addressed by reformulating the Kalman update. The method proposed by Brown and Hwang,[14] which is followed in this paper, derives a new Kalman gain as:

$$K_k = \left(\bar{P}_k H^T + \Phi P_{k-1} J^T\right) \left(H\bar{P}_k H^T + R + J P_{k-1} \Phi^T H^T + H\Phi P_{k-1} J^T + J P_{k-1} J^T\right)^{-1}. \tag{16}$$

The update step becomes

$$\hat{\boldsymbol{x}}_k = \bar{\boldsymbol{x}}_k + K_k \left(\boldsymbol{z}_k - H\bar{\boldsymbol{x}}_k - J\bar{\boldsymbol{x}}_{k-1}\right), \tag{17}$$

$$P_k = \bar{P}_k - K_k L_k K_k^T, \tag{18}$$

where:

$$L_k = H\bar{P}_k H^T + R + J P_{k-1} \Phi^T H^T + H\Phi P_{k-1} J^T + J P_{k-1} J^T. \tag{19}$$

In the case considered, we use the angular velocity of gyro 2 as a measurement, and therefore

$$\boldsymbol{z}_k = \widetilde{\omega}_k^{(2)}, \tag{20}$$

$$H = \frac{1}{\Delta t}, \tag{21}$$

$$J = -\frac{1}{\Delta t}, \tag{22}$$

$$R = \sigma_{(2)}^2. \tag{23}$$

It follows that

$$K_k = \Delta t \frac{\sigma_{(1)}^2}{\sigma_{(1)}^2 + \sigma_{(2)}^2}, \tag{24}$$

$$L_k = \sigma_{(1)}^2 + \sigma_{(2)}^2, \tag{25}$$

and the update step is given by

$$\hat{\theta}_k = \bar{\theta}_k + \Delta t \frac{\sigma_{(1)}^2}{\sigma_{(1)}^2 + \sigma_{(2)}^2} \left(\widetilde{\omega}_k^{(2)} - \widetilde{\omega}_k^{(1)}\right), \tag{26}$$

$$P_k = \bar{P}_k - \Delta t^2 \frac{\sigma_{(1)}^4}{\sigma_{(1)}^2 + \sigma_{(2)}^2}. \tag{27}$$

Substituting Eq. (13), and Eq. (14) into Eq. (26), and Eq. (27), the optimal estimate $\theta_k \sim \mathcal{N}(\hat{\theta}_k, P_k)$ given an initial estimate $\theta_{k-1} \sim \mathcal{N}(\hat{\theta}_{k-1}, P_{k-1})$ and measurements $\omega_k^{(1)}$, $\omega_k^{(2)}$, is

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \frac{\sigma_{(2)}^2 \widetilde{\omega}_k^{(1)} + \sigma_1^2 \widetilde{\omega}_k^{(2)}}{\sigma_{(1)}^2 + \sigma_{(2)}^2}, \tag{28}$$

$$\hat{P}_k = \hat{P}_{k-1} + \Delta t^2 \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{\sigma_{(1)}^2 + \sigma_{(2)}^2}. \tag{29}$$

Time Update via Gyroscope Measurement Averaging

The previous section proposed an optimal method to propagate an angular location estimate using two uncorrelated gyro measurements. This section demonstrates that averaging the gyro measurements leads to the same outcome.

Consider the weighted average angular rate:

$$\widetilde{\omega}_k^{(*)} = \frac{\sum_i \frac{1}{\sigma_{(i)}^2} \widetilde{\omega}_k^{(i)}}{\sum_i \frac{1}{\sigma_{(i)}^2}} \tag{30}$$

$$= \frac{\sigma_{(2)}^2 \widetilde{\omega}_k^{(1)} + \sigma_{(1)}^2 \widetilde{\omega}_k^{(2)}}{\sigma_{(1)}^2 + \sigma_{(2)}^2}. \tag{31}$$

It can be proven that

$$\widetilde{\omega}_k^{(*)} \sim \mathcal{N}\left(\omega_k, \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{\sigma_{(1)}^2 + \sigma_{(2)}^2}\right). \tag{32}$$

The angular location can be propagated from time $k-1$ to time k using the weighted angular velocity $\widetilde{\omega}_k^{(*)}$ as input in Equations (7),(8). The state estimate at time k is $\theta_k \sim \mathcal{N}(\hat{\theta}_k, P_k)$ where:

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \frac{\sigma_{(2)}^2 \widetilde{\omega}_k^{(1)} + \sigma_1^2 \widetilde{\omega}_k^{(2)}}{\sigma_{(1)}^2 + \sigma_{(2)}^2}, \tag{33}$$

$$\hat{P}_k = \hat{P}_{k-1} + \Delta t^2 \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{\sigma_{(1)}^2 + \sigma_{(2)}^2}. \tag{34}$$

This demonstrates that propagating the angular location using the weighted angular rate yields the same optimal estimate as propagating with one gyro measurement and updating with the other.

Angular Velocity Model Integration and Dual Updates

In many aerospace applications, angular velocity can be predicted by integrating a dynamical model. For spacecraft, this model is governed by the Euler equations.[6] For rovers, angular velocity can be determined from odometry measurements. Assume we have a model that provides an unbiased estimate $\hat{\omega}_k$ of the angular rate:

$$\hat{\omega}_k = \omega_k + \nu_m, \tag{35}$$

$$\nu_m \sim \mathcal{N}\left(0,\, \sigma_m^2\right). \tag{36}$$

We want to use the angular rate coming from the model $\hat{\omega}_k$ to propagate the state and the angular rates from the gyros $\widetilde{\omega}_k^{(1)}$, $\widetilde{\omega}_k^{(2)}$ to update the state. This method creates a framework in which dealing with correlated gyro is straightforward.

Similarly to what did before, it is possible to propagate the attitude from time $k-1$ to time k using the angular velocity model $\hat{\omega}_k$. The result of the propagation is $\theta_k \sim \mathcal{N}(\bar{\theta}_k, \bar{P}_k)$ where:

$$\bar{\theta}_k = \hat{\theta}_{k-1} + \hat{\omega}_k \Delta t, \tag{37}$$

$$\bar{P}_k = P_{k-1} + \Delta t^2 \sigma_m^2. \tag{38}$$

This estimate can be updated using data from gyro 1 and gyro 2, along with the delayed-state Kalman update applied earlier. In this case

$$H = \begin{bmatrix} \frac{1}{\Delta t} \\ \frac{1}{\Delta t} \end{bmatrix}, \tag{39}$$

$$J = \begin{bmatrix} -\frac{1}{\Delta t} \\ -\frac{1}{\Delta t} \end{bmatrix}, \tag{40}$$

$$R = \begin{bmatrix} \sigma_{(1)}^2 & 0 \\ 0 & \sigma_{(2)}^2 \end{bmatrix}. \tag{41}$$

It follows that

$$K_k = \Delta t \begin{bmatrix} \frac{\sigma_{(2)}^2 \sigma_m^2}{\sigma_{(1)}^2 \sigma_{(2)}^2 + \sigma_{(1)}^2 \sigma_m^2 + \sigma_{(2)}^2 \sigma_m^2} & \frac{\sigma_{(1)}^2 \sigma_m^2}{\sigma_{(1)}^2 \sigma_{(2)}^2 + \sigma_{(1)}^2 \sigma_m^2 + \sigma_{(2)}^2 \sigma_m^2} \end{bmatrix}, \tag{42}$$

$$L_k = \begin{bmatrix} \sigma_{(1)}^2 + \sigma_m^2 & \sigma_m^2 \\ \sigma_m^2 & \sigma_{(2)}^2 + \sigma_m^2 \end{bmatrix}. \tag{43}$$

After the update step, the optimal estimate at time k, given a model for the angular velocity $\hat{\omega}_k$, and uncorrelated measurements form distinct Gyros $\widetilde{\omega}_k^{(1)}$, $\widetilde{\omega}_k^{(2)}$, is

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \frac{\sigma_{(1)}^2 \sigma_{(2)}^2 \hat{\omega}_k + \sigma_{(2)}^2 \sigma_m^2 \widetilde{\omega}_k^{(1)} + \sigma_{(1)}^2 \sigma_m^2 \widetilde{\omega}_k^{(2)}}{\sigma_{(1)}^2 \sigma_{(2)}^2 + \sigma_{(1)}^2 \sigma_m^2 + \sigma_{(2)}^2 \sigma_m^2}, \tag{44}$$

$$P_k = P_{k-1} + \Delta t^2 \frac{\sigma_{(1)}^2 \sigma_{(2)}^2 \sigma_m^2}{\sigma_{(1)}^2 \sigma_{(2)}^2 + \sigma_{(1)}^2 \sigma_m^2 + \sigma_{(2)}^2 \sigma_m^2}. \tag{45}$$

If the angular velocity model is unknown, i.e. $\sigma_m^2 \to \infty$, the estimated angular location is the same as the one obtained only by propagating with an average angular rate.

$$\lim_{\sigma_m^2 \to \infty} \hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \frac{\sigma_{(2)}^2 \widetilde{\omega}_k^{(1)} + \sigma_{(1)}^2 \widetilde{\omega}_k^{(2)}}{\sigma_{(1)}^2 + \sigma_{(2)}^2}, \tag{46}$$

$$\lim_{\sigma_m^2 \to \infty} \hat{P}_k = \hat{P}_{k-1} + \Delta t^2 \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{\sigma_{(1)}^2 + \sigma_{(2)}^2}. \tag{47}$$

This approach provides a procedure to obtain optimal estimates when the gyro measurements are correlated. The modeled angular velocity can be used to propagate the attitude, while the measurements $\widetilde{\omega}_k^{(1)}$, $\widetilde{\omega}_k^{(2)}$ to update the estimate. In this case, the covariance matrix is

$$R = \begin{bmatrix} \sigma_{(1)}^2 & \sigma_{(1,2)}^2 \\ \sigma_{(1,2)}^2 & \sigma_{(2)}^2 \end{bmatrix}. \tag{48}$$

If the modeled angular velocity is unknown, i.e. $\sigma_m^2 \to \infty$, it is still possible to obtain the optimal estimate in the case in which the gyros are correlated and no other information is available. After some algebraic manipulation, it is possible to obtain

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \frac{\left(\sigma_{(2)}^2 - \sigma_{(1,2)}^2\right)\widetilde{\omega}_k^{(1)} + \left(\sigma_{(1)}^2 - \sigma_{(1,2)}^2\right)\widetilde{\omega}_k^{(2)}}{\sigma_{(1)}^2 + \sigma_{(2)}^2 - 2\sigma_{(1,2)}^2}, \tag{49}$$

$$P_k = P_{k-1} + \Delta t^2 \frac{\sigma_{(1)}^2\sigma_{(2)}^2 - \sigma_{(1,2)}^4}{\sigma_{(1)}^2 + \sigma_{(2)}^2 - 2\sigma_{(1,2)}^2}. \tag{50}$$

Optimal Averaging of Correlated Gyros

This section provides a formal proof of the optimality of the estimates given by Equations (49), (50). Optimality, in this context, implies that the estimate is unbiased and that its covariance is minimized.

Consider propagating the attitude using a linear combination of angular velocity measurements, which may be correlated, namely $\mathbb{E}[\nu_k^{(1)}\nu_k^{(2)}] = \sigma_{(1,2)}^2$

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \left(\alpha\,\widetilde{\omega}_k^{(1)} + \beta\,\widetilde{\omega}_k^{(2)}\right). \tag{51}$$

The first condition for optimality is that the estimator is unbiased:

$$\theta_{k-1} + \Delta t\,\omega_k = \mathbb{E}\left[\hat{\theta}_k\right] \tag{52}$$

$$= \mathbb{E}\left[\hat{\theta}_{k-1} + \Delta t \left(\alpha\,\widetilde{\omega}_k^{(1)} + \beta\,\widetilde{\omega}_k^{(2)}\right)\right] \tag{53}$$

$$= \theta_{k-1} + \Delta t\ (\alpha + \beta)\ \omega_k, \tag{54}$$

therefore

$$\alpha + \beta = 1, \tag{55}$$

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \left(\widetilde{\omega}_k^{(1)} + \beta\left(\widetilde{\omega}_k^{(2)} - \widetilde{\omega}_k^{(1)}\right)\right). \tag{56}$$

The second condition is that the covariance is minimized:

$$P_k = P_{k-1} + \Delta t^2 \left[\sigma_{(1)}^2 + \beta^2(\sigma_{(1)}^2 + \sigma_{(2)}^2) - 2\beta\sigma_{(1)}^2 + 2\beta(1-\beta)\sigma_{(1,2)}^2\right], \tag{57}$$

and applying the first optimality condition

$$\frac{\partial P_k}{\partial \beta} = 0 \quad \Rightarrow \quad \beta = \frac{\sigma_{(1)}^2 - \sigma_{(1,2)}^2}{\sigma_{(1)}^2 + \sigma_{(2)}^2 - 2\sigma_{(1,2)}^2}. \tag{58}$$

Therefore after some substitutions, we proved that the optimal estimate is given by

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \Delta t \frac{\left(\sigma_{(2)}^2 - \sigma_{(1,2)}^2\right) \widetilde{\omega}_k^{(1)} + \left(\sigma_{(1)}^2 - \sigma_{(1,2)}^2\right) \widetilde{\omega}_k^{(2)}}{\sigma_{(1)}^2 + \sigma_{(2)}^2 - 2\sigma_{(1,2)}^2}, \tag{59}$$

$$P_k = P_{k-1} + \Delta t^2 \frac{\sigma_{(1)}^2 \sigma_{(2)}^2 - \sigma_{(1,2)}^4}{\sigma_{(1)}^2 + \sigma_{(2)}^2 - 2\sigma_{(1,2)}^2}. \tag{60}$$

For a general case, in which measurements from N gyros are available, and the measurement can be correlated, we can get the optimal estimate by averaging the angular rates

$$R = \begin{bmatrix} \sigma_{(1)}^2 & \sigma_{(1,2)}^2 & \cdots & \sigma_{(1,N)}^2 \\ \sigma_{(1,2)}^2 & \sigma_{(2)}^2 & \cdots & \sigma_{(2,N)}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{(1,N)}^2 & \sigma_{(2,N)}^2 & \cdots & \sigma_{(N)}^2 \end{bmatrix}, \tag{61}$$

$$\Gamma = R^{-1}, \tag{62}$$

$$\widetilde{\omega}_k^{(*)} = \frac{\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij}\, \widetilde{\omega}_k^{(i)}}{\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij}}, \tag{63}$$

$$\sigma_{(*)}^2 = \frac{1}{\sum_{i=1}^N \sum_{j=1}^N \gamma_{ij}}, \tag{64}$$

$$\widetilde{\omega}_k^{(*)} \sim \mathcal{N}\left(\omega_k, \sigma_{(*)}^2\right) \tag{65}$$

and using this angular rate to propagate the estimate from time $k-1$ to time k.

PLANAR ROTATION WITH BIASED GYRO MEASUREMENTS

The previous section showed how to use measurements coming from bias-free gyros to propagate the angle in an optimal way. However, to apply this approach to practical applications, biases must be incorporated into the measurement model for the gyroscopes. This section introduces a method for averaging measurements from biased gyros under specific assumptions.

Consider measurements coming from two uncorrelated gyros

$$\widetilde{\omega}_k^{(i)} = \frac{1}{\Delta t}\left(\theta_k - \theta_{k-1}\right) + b_k^{(i)} + \nu_k^{(i)}, \quad i = 1, 2, \tag{66}$$

where $b_k^{(i)}$ is the bias of the *i-th* gyro at time k. The bias drifts over time due to the bias instability. A mathematical model widely used to describe the evolution in time of the bias is given by[6]

$$\dot{b}^{(i)}(t) = \eta_b^{(i)}(t), \tag{67}$$

where $\eta_b^{(i)}(t)$ is a zero-mean Gaussian white-noise process. Under the assumption of a fast enough sampling rate, Equation (67) can be discretized considering $\eta_b^{(i)}(t)$ constant in the interval $k-1$ to k. It follows

$$b_k^{(i)} = b_{k-1}^{(i)} + \nu_k^{(i),b} \tag{68}$$

where $\nu_k^{(i),b}$ is a zero-mean Gaussian random sequence, i.e. $\nu_k^{(i),b} \sim \mathcal{N}(0, \sigma^2_{(i),b})$.

The estimated state is given by

$$\boldsymbol{x}_k = \begin{bmatrix} \theta_k \\ b_k^{(1)} \\ b_k^{(2)} \end{bmatrix} \tag{69}$$

The problem involves estimating the angle θ_k and the biases $b_k^{(1)}$, $b_k^{(2)}$, given the measurements $\widetilde{\omega}_k^{(1)}$ and $\widetilde{\omega}_k^{(2)}$, as well as an estimate of the state at the previous time $\boldsymbol{x}_{k-1} \sim \mathcal{N}(\hat{\boldsymbol{x}}_{k-1}, P_{k-1})$.

In the remainder of the paper, we will adopt the simplification that the gyros are similar, meaning that their noise characteristics are in a constant ratio; i.e., the following relationship holds

$$\frac{\sigma^2_{(1)}}{\sigma^2_{(2)}} = \frac{\sigma^2_{(1),b}}{\sigma^2_{(2),b}} = \frac{\operatorname{cov}\left(b_{k-1}^{(1)}\right)}{\operatorname{cov}\left(b_{k-1}^{(2)}\right)} = \rho. \tag{70}$$

In practical applications, redundant gyros are typically produced by the same manufacturer, resulting in identical noise specifications, i.e. $\rho = 1$. The following notation is used:

$$\sigma^2 = \sigma^2_{(2)} = \frac{\sigma^2_{(1)}}{\rho}, \tag{71}$$

$$\sigma^2_b = \sigma^2_{(2),b} = \frac{\sigma^2_{(1),b}}{\rho}. \tag{72}$$

Time and Measurement Updates Using Distinct Gyroscopes

Similarly to what was done in the bias free case, it is possible to use the measurement coming from gyro 1 as input to propagate the state, and the measurement coming from gyro 2 to perform the measurement update. For a linear system subjected to an uncertain input $\boldsymbol{u}_{k-1} \sim \mathcal{N}(\hat{\boldsymbol{u}}_{k-1}, U)$ and some process noise $\boldsymbol{v}_{k-1} \sim \mathcal{N}(\boldsymbol{0}, Q)$, the dynamics are:

$$\bar{\boldsymbol{x}}_k = \Phi \hat{\boldsymbol{x}}_{k-1} + B\hat{\boldsymbol{u}}_{k-1}, \tag{73}$$

$$\bar{P}_k = \Phi P_{k-1} \Phi^T + BUB^T + \Gamma Q \Gamma^T. \tag{74}$$

For a planar rotation with biased angular velocity measurements, it follows that

$$\hat{\boldsymbol{u}}_{k-1} = \widetilde{\omega}_k^{(1)}, \tag{75}$$

$$\Phi = \begin{bmatrix} 1 & -\Delta t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{76}$$

$$B = \begin{bmatrix} \Delta t \\ 0 \\ 0 \end{bmatrix}, \tag{77}$$

$$\Gamma = \begin{bmatrix} -\Delta t & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{78}$$

$$U = \rho \sigma^2, \tag{79}$$

$$Q = \begin{bmatrix} \rho \sigma_b^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}. \tag{80}$$

To perform the measurement update using the angular velocity coming from gyro 2, the modified Kalman update described in Equations (15)-(19) is used. In the case considered, it is evident that:

$$H = \begin{bmatrix} \frac{1}{\Delta t} & 0 & 1 \end{bmatrix}, \tag{81}$$

$$J = \begin{bmatrix} -\frac{1}{\Delta t} & 0 & 0 \end{bmatrix}, \tag{82}$$

$$R = \sigma^2. \tag{83}$$

Considering an initial state $\boldsymbol{x}_{k-1} \sim \mathcal{N}(\hat{\boldsymbol{x}}_{k-1}, P_{k-1})$, where

$$\hat{\boldsymbol{x}}_{k-1} = \begin{bmatrix} \hat{\theta}_{k-1} \\ \hat{b}_{k-1}^{(1)} \\ \hat{b}_{k-1}^{(2)} \end{bmatrix}, \quad P_{k-1} = \begin{bmatrix} P_{\theta\theta} & 0 & 0 \\ 0 & \rho P_{bb} & 0 \\ 0 & 0 & P_{bb} \end{bmatrix} \tag{84}$$

upon propagating the state using $\widetilde{\omega}_k^{(1)}$ and updating it using $\widetilde{\omega}_k^{(2)}$, the estimate at time k is given by:

$$\hat{\boldsymbol{x}}_k = \begin{bmatrix} \hat{\theta}_k \\ \hat{b}_k^{(1)} \\ \hat{b}_k^{(2)} \end{bmatrix} = \begin{bmatrix} \hat{\theta}_{k-1} + \Delta t \dfrac{\left(\widetilde{\omega}_k^{(1)} - b_{k-1}^{(1)}\right) + \rho \left(\widetilde{\omega}_k^{(2)} - b_{k-1}^{(2)}\right)}{1+\rho} \\ \hat{b}_{k-1}^{(1)} - \dfrac{\rho \mathrm{B} \left(\widetilde{\omega}_k^{(2)} - b_{k-1}^{(2)} - \widetilde{\omega}_k^{(1)} + b_{k-1}^{(1)}\right)}{\mathrm{A}} \\ \hat{b}_{k-1}^{(2)} + \dfrac{\mathrm{B} \left(\widetilde{\omega}_k^{(2)} - b_{k-1}^{(2)} - \widetilde{\omega}_k^{(1)} + b_{k-1}^{(1)}\right)}{\mathrm{A}} \end{bmatrix}, \tag{85}$$

$$P_k = \begin{bmatrix} P_{\theta\theta} + \Delta t^2 \dfrac{P_{bb}\rho + \rho\sigma^2 + \rho\sigma_b^2}{\rho+1} & -\dfrac{\Delta t \rho \mathrm{B}}{\rho+1} & -\dfrac{\Delta t \rho \mathrm{B}}{\rho+1} \\ -\dfrac{\Delta t \rho \mathrm{B}}{\rho+1} & \rho \mathrm{B} - \dfrac{\rho^2 \mathrm{B}^2}{\mathrm{A}} & \dfrac{\rho \mathrm{B}^2}{\mathrm{A}} \\ -\dfrac{\Delta t \rho \mathrm{B}}{\rho+1} & \dfrac{\rho \mathrm{B}^2}{\mathrm{A}} & P_{bb} + \sigma_b^2 - \dfrac{\mathrm{B}^2}{\mathrm{A}} \end{bmatrix}, \tag{86}$$

where

$$\mathrm{A} = (\rho+1)\left(P_{bb} + \sigma^2 + \sigma_b^2\right), \tag{87}$$

$$\mathrm{B} = P_{bb} + \sigma_b^2. \tag{88}$$

The covariance matrix P_k associated with this estimate is a full matrix, meaning that even if the state θ, $b^{(1)}$, and $b^{(2)}$ were initially uncorrelated, after the first update step they become fully correlated.

Transformation of variables. In this section, we propose a linear transformation of both the state and the input. This approach serves as the foundation for a new method introduced in next section of this paper, which incorporates information from the two gyros without relying on the delayed state update.

It is possible to introduce an average bias $b_k^{(*)}$ as

$$b_k^{(*)} = \frac{b_k^{(1)} + \rho b_k^{(2)}}{1+\rho}, \tag{89}$$

and a difference bias Δb_k as

$$\Delta b_k = b_k^{(1)} - b_k^{(2)}. \tag{90}$$

After this linear transformation, the state to be estimated becomes

$$\begin{bmatrix} \theta_k \\ b_k^{(1)} \\ b_k^{(2)} \end{bmatrix} \implies \begin{bmatrix} \theta_k \\ b_k^{(*)} = \dfrac{b_k^{(1)} + \rho b_k^{(2)}}{1+\rho} \\ \Delta b_k = b_k^{(1)} - b_k^{(2)} \end{bmatrix}. \tag{91}$$

Similarly, it is possible to transform the measurements coming from the two gyros. We introduce an average angular velocity measurement $\widetilde{\omega}_k^{(*)}$ as

$$\widetilde{\omega}_k^{(*)} = \frac{\widetilde{\omega}_k^{(1)} + \rho \widetilde{\omega}_k^{(2)}}{1+\rho}, \tag{92}$$

and a difference angular velocity measurement $\Delta\widetilde{\omega}_k$ as

$$\Delta\widetilde{\omega}_k = \widetilde{\omega}_k^{(1)} - \widetilde{\omega}_k^{(2)}. \tag{93}$$

After applying these transformations to the state and input, the estimated state given by Equations (85), (86) becomes:

$$\hat{\boldsymbol{x}}_k = \begin{bmatrix} \hat{\theta}_k \\ \hat{b}_k^{(*)} \\ \Delta\hat{b}_k \end{bmatrix} = \begin{bmatrix} \hat{\theta}_{k-1} + \Delta t \left(\widetilde{\omega}_k^{(*)} - b_{k-1}^{(*)} \right) \\ \hat{b}_{k-1}^{(*)} \\ \dfrac{\sigma^2 \Delta\hat{b}_{k-1} + \left(P_{bb} + \sigma_b^2\right) \Delta\widetilde{\omega}_k}{\left(P_{bb} + \sigma^2 + \sigma_b^2\right)} \end{bmatrix}, \tag{94}$$

$$P_k = \begin{bmatrix} P_{\theta\theta} + \Delta t^2 \dfrac{P_{bb}\rho + \rho\sigma^2 + \rho\sigma_b^2}{\rho+1} & -\dfrac{\Delta t \rho \left(P_{bb} + \sigma_b^2\right)}{\rho+1} & 0 \\ -\dfrac{\Delta t \rho \left(P_{bb} + \sigma_b^2\right)}{\rho+1} & \dfrac{\rho \left(P_{bb} + \sigma_b^2\right)}{\rho+1} & 0 \\ 0 & 0 & \dfrac{\sigma \left(P_{bb} + \sigma_b^2\right)(\rho+1)}{P_{bb} + \sigma^2 + \sigma_b^2} \end{bmatrix}. \tag{95}$$

These transformations highlight the fact that the attitude is propagated optimally by considering only the average angular rate and the average bias. Furthermore, the expected value of the average bias is constant, while its covariance increases over time. The difference in biases depends solely on the difference in angular rates. It is also important to note that the difference in biases is uncorrelated from both the average bias and the attitude.

Time Update via Gyroscope Measurement Averaging

In the previous section, we demonstrated that it is possible to propagate the attitude optimally using only the average gyro measurement and the average bias. Building on this result, in this section, we develop a new approach to estimate the angular rotation and the gyro biases without requiring the delayed state Kalman update.

First, reconsider the linear transformation of the state described by Equations (89), (90). The following dynamics for the biases hold:

$$b_k^{(*)} = b_{k-1}^{(*)} + \nu_k^{(*),b}, \tag{96}$$

$$\Delta b_k = \Delta b_{k-1} + \nu_k^{\Delta b}, \tag{97}$$

where $\nu_k^{(*),b} \sim \mathcal{N}\left(0, \frac{\rho}{1+\rho}\sigma_b^2\right)$ and $\nu_k^{\Delta b} \sim \mathcal{N}\left(0, (1+\rho)\,\sigma_b^2\right)$. Furthermore, Equation (94) shows that the angle θ can be propagated as:

$$\theta_{k-1} = \theta_{k-1} + \Delta t\left(\widetilde{\omega}_k^{(*)} - b_k^{(*)}\right), \tag{98}$$

where $\widetilde{\omega}_k^{(*)} \sim \mathcal{N}\left(\omega_k, \frac{\rho}{1+\rho}\sigma^2\right)$

Therefore, it is possible to perform a time update step starting from the estimate at time $k-1$. It is important to note that, due to the linear transformation of the state, the initial covariance matrix in this case is:

$$P_{k-1} = \begin{bmatrix} P_{\theta\theta} & 0 & 0 \\ 0 & \frac{\rho}{1+\rho}P_{bb} & 0 \\ 0 & 0 & P_{bb}\,(1+\rho) \end{bmatrix}. \tag{99}$$

Utilizing Equations (73), (74), and remembering that in this case

$$\hat{\boldsymbol{u}}_{k-1} = \widetilde{\omega}_k^{(*)}, \tag{100}$$

$$\Phi = \begin{bmatrix} 1 & -\Delta t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{101}$$

$$B = \begin{bmatrix} \Delta t \\ 0 \\ 0 \end{bmatrix}, \tag{102}$$

$$\Gamma = \begin{bmatrix} -\Delta t & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{103}$$

$$U = \frac{\rho}{1+\rho}\sigma^2, \tag{104}$$

$$Q = \begin{bmatrix} \frac{\rho}{1+\rho}\sigma_b^2 & 0 \\ 0 & (1+\rho)\,\sigma_b^2 \end{bmatrix}, \tag{105}$$

the prior estimate at time k is:

$$\bar{\boldsymbol{x}}_k = \begin{bmatrix} \hat{\theta}_k \\ \hat{b}_k^{(*)} \\ \Delta\hat{b}_k \end{bmatrix} = \begin{bmatrix} \hat{\theta}_{k-1} + \Delta t \left(\widetilde{\omega}_k^{(*)} - b_{k-1}^{(*)} \right) \\ \hat{b}_{k-1}^{(*)} \\ \Delta\hat{b}_{k-1} \end{bmatrix}, \tag{106}$$

$$\bar{P}_k = \begin{bmatrix} P_{\theta\theta} + \Delta t^2 \frac{P_{bb}\rho + \rho\sigma^2 + \rho\sigma_b^2}{\rho + 1} & -\frac{\Delta t \rho \left(P_{bb} + \sigma_b^2 \right)}{\rho + 1} & 0 \\ -\frac{\Delta t \rho \left(P_{bb} + \sigma_b^2 \right)}{\rho + 1} & \frac{\rho \left(P_{bb} + \sigma_b^2 \right)}{\rho + 1} & 0 \\ 0 & 0 & \left(P_{bb} + \sigma_b^2 \right) (\rho + 1) \end{bmatrix}. \tag{107}$$

By comparing Equations (94),(95) with Equations (106),(107), it is clear that while the expected values and covariances of the angular location and the average bias are identical between the two proposed methods, the estimate of the difference in biases is not. This discrepancy arises because we are only using a linear combination of the two inputs and are neglecting some information that could be obtained by considering each individual input.

Measurement Update via Gyroscope Measurement Difference. The method we propose involves updating the state using the difference in the angular velocities of the gyros, after propagating the state with the average angular rate. Equations (108)-(111) show that the difference in the angular velocities of the gyros directly corresponds to the difference in biases. Therefore, it is possible to perform a classical Kalman update since the measurement depends solely on the state at time k.

$$\begin{aligned} \Delta\widetilde{\omega}_k &= \widetilde{\omega}_k^{(1)} - \widetilde{\omega}_k^{(2)} && (108) \\ &= \widetilde{\omega}_k + b_k^{(1)} + \nu^{(1)} - \widetilde{\omega}_k - b_k^{(2)} - \nu^{(2)} && (109) \\ &= b_k^{(1)} - b_k^{(2)} - \nu^{(2)} + \nu^{(1)} && (110) \\ &= \Delta b_k + \nu_k^{\Delta} && (111) \end{aligned}$$

where $\nu_k^{\Delta} \sim \mathcal{N}\left(0, (1 + \rho)\, \sigma^2\right)$.

Considering the measurement function:

$$\boldsymbol{z}_k = H\boldsymbol{x}_k + \boldsymbol{v}_k, \tag{112}$$

where $\boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, R)$ and

$$H = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \tag{113}$$

$$R = (1 + \rho)\, \sigma^2 \tag{114}$$

and considering the Kalman update:

$$\begin{aligned} K_k &= \left(\bar{P}_k H^T \right) \left(H \bar{P}_k H^T + R \right)^{-1}, && (115) \\ \hat{\boldsymbol{x}}_k &= \bar{\boldsymbol{x}}_k + K_k \left(\boldsymbol{z}_k - H\bar{\boldsymbol{x}}_k \right), && (116) \\ P_k &= \bar{P}_k - K_k H \bar{P}_k. && (117) \end{aligned}$$

It follows that, after the update, the estimated state is:

$$\hat{\boldsymbol{x}}_k = \begin{bmatrix} \hat{\theta}_k \\ \hat{b}_k^{(*)} \\ \Delta\hat{b}_k \end{bmatrix} = \begin{bmatrix} \hat{\theta}_{k-1} + \Delta t \left(\widetilde{\omega}_k^{(*)} - b_{k-1}^{(*)} \right) \\ \hat{b}_{k-1}^{(*)} \\ \dfrac{\sigma^2 \Delta\hat{b}_{k-1} + \left(P_{bb} + \sigma_b^2\right) \Delta\widetilde{\omega}_k}{\left(P_{bb} + \sigma^2 + \sigma_b^2\right)} \end{bmatrix}, \tag{118}$$

$$P_k = \begin{bmatrix} P_{\theta\theta} + \Delta t^2 \dfrac{P_{bb}\rho + \rho\sigma^2 + \rho\sigma_b^2}{\rho+1} & -\dfrac{\Delta t \rho \left(P_{bb} + \sigma_b^2\right)}{\rho+1} & 0 \\ -\dfrac{\Delta t \rho \left(P_{bb} + \sigma_b^2\right)}{\rho+1} & \dfrac{\rho \left(P_{bb} + \sigma_b^2\right)}{\rho+1} & 0 \\ 0 & 0 & \dfrac{\sigma \left(P_{bb} + \sigma_b^2\right)(\rho+1)}{P_{bb} + \sigma^2 + \sigma_b^2} \end{bmatrix}. \tag{119}$$

By comparing Equations (94),(95) with Equations (118),(119), it is clear that the estimated state and covariance are the same. We have thus proven that propagating the state using the measurement from one gyro and updating the state with the measurement from the other gyro, while employing the delayed state update, is equivalent to propagating the state with the average measurement $\widetilde{\omega}^{(*)}$ and performing a standard Kalman update using the difference measurement $\Delta\widetilde{\omega}$.

Furthermore, we demonstrated that, through the linear transformation of the biases, the difference in biases Δb is uncorrelated from both the average bias $b^{(*)}$ and the angle θ. This implies that to propagate the attitude optimally, only the propagation step with $\omega^{(*)}$ is necessary. The update step makes the difference in biases observable and is only useful if we are interested in determining the individual biases, for example for fault detection and isolation purposes.

Observability Analysis and Attitude Measurement. Consider a generic linear time-invariant system in discrete time

$$\boldsymbol{x}_{k+1} = \Phi \boldsymbol{x}_k + B\boldsymbol{u}_k, \tag{120}$$

$$\boldsymbol{z}_k = H\boldsymbol{x}_k + D\boldsymbol{u}_k. \tag{121}$$

$$\tag{122}$$

The observability matrix can be computed as:

$$\mathcal{O} = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \\ H\Phi^{n-1} \end{bmatrix}, \tag{123}$$

where n is the number of states.

In the proposed method, the Φ and H matrices are defined respectively in Equations (101), (113). Therefore, the observability matrix becomes:

$$\mathcal{O} = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{124}$$

It is evident that this matrix has rank 1, as only the difference in biases Δb is observable due to its uncorrelation with the other states $b^{(*)}$, θ.

To make the system fully observable, a measurement of the angular location $\widetilde{\theta}_k$ is introduced. The measurement model in this case is:

$$\widetilde{\theta}_k = \theta_k + \nu_k^\theta, \tag{125}$$

where $\nu_k^\theta \sim \mathcal{N}\left(0, \sigma_\theta^2\right)$

It is clear that the state can be updated using both the angular measurement and the difference in angular velocity. The complete measurement model is:

$$\boldsymbol{z}_k = \begin{bmatrix} \tilde{\theta}_k \\ \Delta\tilde{\omega}_k \end{bmatrix} = H\boldsymbol{x}_k + \boldsymbol{v}_k, \tag{126}$$

where $\boldsymbol{v}_k \sim \mathcal{N}(\mathbf{0}, R)$ and

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{127}$$

$$R = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & (1+\rho)\,\sigma^2 \end{bmatrix}. \tag{128}$$

In this case, the observability matrix can be recomputed, yielding:

$$\mathcal{O} = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -\Delta t & 0 \\ 0 & 0 & 1 \\ 1 & -2\Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{129}$$

It is evident that, under the basic assumption of $\Delta t \neq 0$, the observability matrix is full rank, meaning all the states are observable.

RESULTS AND DISCUSSION

To evaluate the performance of the proposed method, a Monte Carlo simulation with 1,000 runs was conducted. We mathematically proved that the two proposed methods are equivalent; therefore, this simulation provides insight into the observability of the states and allows for a performance comparison against the current state-of-the-art approach, which relies on using only a single gyro. This section is organized as follows: first, the Monte Carlo simulation parameters are described, including the specifications of each sensor and the method used to generate the truth trajectory. Finally, the numerical results are presented and discussed.

Monte Carlo Settings

The simulated scenario involves a lunar rover estimating its heading angle, θ, with respect to an inertial reference frame using angular measurements from two baised gyros and a star tracker. A star tracker provides two angle measurements: the heading and the elevation of a star. Since this application is limited to planar motion, the elevation angle is discarded.

For the simulation of the truth, we imposed that the true angular velocity follows a cosinusoidal function:

$$\omega(t) = \omega_0 \cos\left(f_0 t\right). \tag{130}$$

The angular velocity can be analytically integrated to determine the true time history of the heading angle over time:

$$\theta(t) = \theta_0 + \frac{\omega_0}{f_0} \sin\left(f_0 t\right). \tag{131}$$

Table 1 reports the parameters used to simulate this true trajectory.

Table 1. Initial conditions of numerical integration.

Parameter	Unit	Value
ω_0	deg/s	5
f_0	Hz	0.1
θ_0	deg	45

The numerical integration outputs were sampled at a fixed interval Δt_G, corresponding to the sampling rate of the gyros. The angular velocity measurements were then corrupted with a random noise sequence to simulate realistic gyro readings. Both gyros were assumed to have identical noise characteristics, i.e., $\rho = 1$.

Regarding the evolution of the gyro biases, although their true dynamics evolve continuously, the following discrete-time approximation was applied for the truth simulation.

$$b_k^{(i)} = b_{k-1}^{(i)} + \nu_b, \tag{132}$$

where $\nu_b \sim \mathcal{N}(0, \sigma_b^2)$. The noise characteristics of the gyros are summarized in Table 2.

Table 2. Gyroscopes parameters.

Parameter	Unit	Value
Δt_G	s	0.01
ARW[15]	deg/h/$\sqrt{Hz}$	0.72
σ_b[15]	deg/h	0.05

Star tracker measurements were simulated by sampling the true heading angle at fixed intervals Δt_{ST} and corrupting them with random noise. The characteristics of the star tracker are provided in Table 3.

Finally, the filters were initialized with the same initial state estimate for each Monte Carlo run, using the covariance matrix given in Table 4.

Monte Carlo Results

The estimation errors obtained from the Monte Carlo analysis are shown in Figures 1. These plots demonstrate that the proposed method is consistent, as the mean of the estimation error remains

Table 3. Star Tracker parameters.

Parameter	Unit	Value
Δt_{ST}	s	1
σ_{ST}	deg	0.01

Table 4. Initial covariance matrix.

Parameter	Unit	Value
$P_{\theta\theta}$	deg^2	$0.1^2\boldsymbol{I}_{3\times 3}$
P_{bb}	deg^2/s^2	$0.01^2\boldsymbol{I}_{3\times 3}$

zero and the true and filter covariances align perfectly. This is expected since both the measurement function and the dynamics are linear, making the Kalman Filter an optimal estimator in this case.

Figures 1(a) and 1(b) show the results obtained by propagating the state with $\widetilde{\omega}^{(*)}$ and updating it with $\Delta\widetilde{\omega}$. In particular, Fig. 1(b) clearly shows that the heading angle θ and the average bias $b^{(*)}$ are unobservable, while the difference in bias Δb is observable.

Figure 1(c) presents the same results as Figure 1(b), but in this case, the state is also updated with the star tracker measurement. Interestingly, while the heading angle and the difference in bias become observable, no observability is gained for the average bias. This is because the difference in bias is uncorrelated with both the heading angle and the average bias. This conclusion is further confirmed by Figure 1(d), which shows results obtained by propagating the state with $\widetilde{\omega}^{(*)}$ and updating it with the star tracker measurement. Comparing Figures 1(c) and 1(d) makes it evident that, without the $\Delta\widetilde{\omega}$ update, the heading angle and the average bias do not lose any observability, but the difference in bias becomes unobservable. This clearly supports the main point of this paper: the estimation of the difference in bias is completely independent from the other components of the state and can be performed when accurate knowledge of the individual biases is necessary.

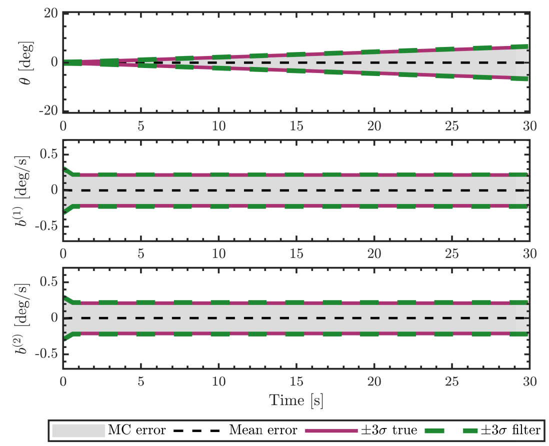
Finally, to evaluate the accuracy of the proposed method, the time-averaged RMSE was computed as

$$\text{RMSE} = \frac{1}{N_t N_m} \sum_{k=1}^{N_t} \sum_{j=1}^{N_m} \sqrt{\sum_{i=1}^{N_s} \frac{(x_{k,j}^{(i)} - \hat{x}_{k,j}^{(i)})^2}{N_s}}. \tag{133}$$

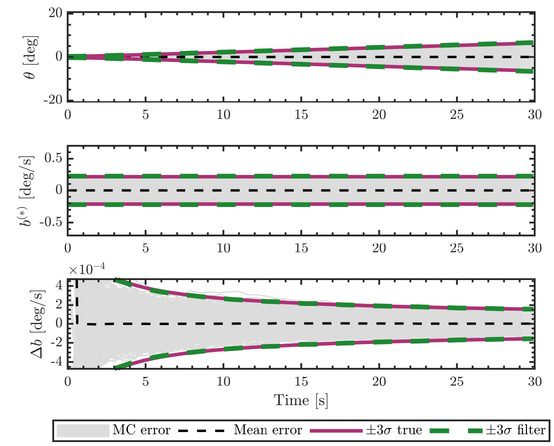
The results are summarized in Table 5, which indicates that utilizing all available gyros improves attitude estimation accuracy by approximately 4% and bias estimation accuracy by approximately 20%.

Table 5. Time-averaged Root Mean Squared Error for the proposed method and using only one gyroscope.

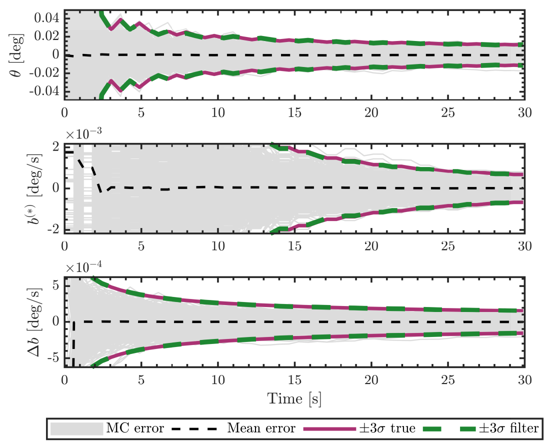
State	Averaging	Single Gyro
θ [arcsec]	29.567	30.724
$b^{(1)}$ [deg/h]	16.34	19.667



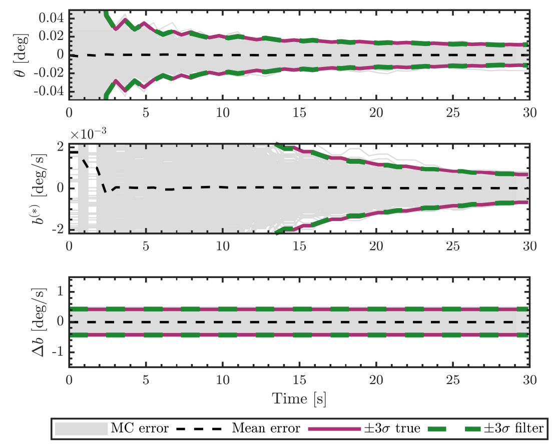
(a) Estimation error of original state as a function of time without angular measurement.



(b) Estimation error of transformed state as a function of time without angular measurement.



(c) Estimation error of transformed state as a function of time with angular measurement.



(d) Estimation error of transformed state as a function of time with angular measurement and without Δb update.

Figure 1. Comparison of estimation errors for different scenarios.

CONCLUSIONS

This work proposes two methods for fusing all available gyroscope measurements when redundant IMUs provide faultless angular velocity estimates. The first approach uses one gyro for state propagation and the other for state updates. The second method employs an averaged measurement in the propagation step while retaining full observability of individual biases by updating the state with the difference in angular velocity measurements. For planar rotation, the two approaches have been shown to be analytically equivalent. Monte Carlo simulations demonstrate that the update step in the second method can be neglected when bias estimates are not critical, without compromising heading angle estimation. Furthermore, incorporating all available measurements improves the estimation accuracy of both attitude and gyro bias.

The primary limitation of the averaging-based filter is its reliance on the assumption that gyroscopes have similar noise characteristics. However, in practice, this is not a significant concern, as redundant IMUs typically share the same noise specifications. While this study focused on two gyroscopes, the proposed algorithms can be readily extended to accommodate additional sensors. Future work includes extending the methods to three-axis rotation and evaluating their performance in a spacecraft attitude control scenario.

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