# Attracting Manifolds for Attitude Estimation in Flatland and Otherlands*

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**Abstract**

Non-convex and non-affine parameterizations of uncertainty are intrinsic within every attitude estimation problem given the fact that minimal and/or nonsingular representations of the attitude matrix are invariably nonlinear functions of the unknown attitude variables. Of course, this fact remains true for rotation matrices both in the 2-D plane (flatland) and in higher dimensional spaces (otherlands). Therefore, estimation problems involving minimal nonsingular representations of unknown attitude matrices bring significant challenges to the adaptive estimation community. This paper develops a novel algorithm for attitude estimation. The proposed algorithm relies upon the design of an adaptive update law for the attitude estimate while preserving its inherent orthogonal structure. The underlying approach borrows from the classical Poisson differential equation in rigid-body rotational kinematics and endows certain manifold attractivity features within the adaptive estimation algorithm. Consequently, we are not only able to efficiently handle the non-affine and non-convex nature of the parameter uncertainty, but are also ensured of estimation algorithm stability and robustness under bounded measurement noise. In addition to a rigorous discussion on the overall methodology, the paper provides example simulations that help demonstrate the effectiveness of the attracting manifolds design.

## 1. Introduction

Attitude estimation problems routinely arise in numerous aerospace engineering and robotics applications. More specifically, relative navigation and attitude determination problems are linearly parameterized by unknown $3\times3$ proper orthogonal matrices SO(3). Numerous attitude estimation/determination algorithms are available in the literature developed by both control and estimation communities. Of immense significance is the fact that every orthogonal matrix is nonlinear in its degrees of freedom[1, 2]. Virtually every existing attitude estimation method converts this nonlinear parameterization into a linear over-parameterization[3]. Depending on how the underlying attitude estimation algorithm is implemented, existing methods can be broadly categorized into two classes: batch type and sequential type estimators.

Batch type estimators such as QUEST[4] and FOAM[5] utilize more than two observations at each observation instant to determine the attitude matrix in three-dimensions, thereby necessitating the use of two or more independent sensors. The underlying estimation is accomplished by minimizing a quadratic cost function originally proposed by Wahba[6]. TRIAD[7] is another variant among existing batch type attitude

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estimation algorithms which requires at most $n-1$ linearly independent observation vectors for attitude determination in an $n$-dimensional space[8].

Sequential type estimators, in contrast with batch type estimators, need only one observation at each time. The most common implementation of sequential estimators adopt an extended Kalman filter[9]. Instead of fusing measurements from multiple sensors at each instant, sequential type estimators utilize an analytic model of the system to forward propagate the observation data. Attitude estimates are then generated by comparing predicted observations with actual measurements while updating the underlying analytic model through minimization of a suitable optimality criterion. The standardly adopted optimality criterion for Kalman filtering is the minimization of variances between estimates from sensor measurement and predicted values derived from the system analytic model. Recent applications of extended Kalman filter type sequential estimators are documented for missions such as the Earth Radiation Budget Satellite (ERBS)[10, 11] and the Solar Anomalous Magnetospheric Particle Explorer (SAMPEX)[12, 13].

Even though batch and sequential type estimators have been successfully applied to a wide array of actual spacecraft missions, both classes of methods usually suffer from a crucial limitation – that of over-parameterization and the resulting non-enforcement of the orthogonality structure on the attitude matrix estimate. In order to eliminate problems associated with over-parameterization, recently, an adaptive algorithm for orthogonal matrix estimation was developed by Kinsey and Whitcomb[14]. The convergence proof for their estimation algorithm, applicable only for attitude estimation in the three-dimensional space, utilizes a matrix logarithmic map defined over the attitude estimation error matrices. Further, this logarithmic map is not injective for certain class of orthogonal matrices. To be precise, this restriction pertains to non-inclusion of SO(3) matrices whose trace is equal to $-1$ (the corresponding Euler principal rotation angle $\phi = \pm\pi$).

This paper aims at deriving new classes of adaptive estimation algorithms for uncertain $n \times n$ proper orthogonal matrices. The proposed estimation methodology involves introduction of an attracting manifold about the "true" but "unknown" attitude matrix which helps in automatically enforcing the attitude matrix estimate to be proper and orthogonal at every time step. Design of the attracting manifold results in significant reduction of computational burden associated with: (a) being able to fully utilize the available prior information on orthogonality of the unknown attitude and thereby avoiding over-parameterization; and (b) not having to perform the re-orthogonalization process at each time step. Additionally, under standard assumptions involving availability of persistence in excitation (PE), we are able to show that the attitude estimate is guaranteed to converge to the corresponding true value. Convergence proof for the estimation algorithm presented in this paper is accomplished in such a way that not only are the restrictions associated with the logarithmic map of Kinsey and Whitcomb[14] are completely eliminated but our results also generalize nicely for attitude matrices on all $n$-dimensional spaces.

The paper is organized as follows. In section 2, we present our main results that establish a new orthogonality preserving attitude matrix estimation algorithm valid for the general $n$-dimensional case. Robustness analysis for this estimation algorithm under the influence of measurement noise is also presented in section 2. Implications of the proposed attitude estimation algorithm for two-dimensions (flatland case) and three-dimensions (otherlands) are discussed in section 3 and section 4 respectively. Numerical simulations are presented in section 5 to demonstrate and validate the various technical claims of this paper. Section 6 provides a discussion with concluding remarks.

# 2. Problem Statement and Main Result

## 2.1 Problem Definition

Succinctly stated, the attitude estimation problem is that of finding a direction cosine matrix describing the orientation of any body fixed frame with respect to the inertial frame. In order to formulate this problem within an analytical framework, we consider estimation of an unknown and time-varying $n \times n$

proper orthogonal matrix $C(t)$ (i.e., satisfying $C(t)C^T(t) = C^T(t)C(t) = I_{n \times n}$ and $\det(C(t))$=1 for all $t$) defined through the following continuous-time input-output mapping

$$\boldsymbol{y}(t) = C(t)\boldsymbol{r}(t) \tag{1}$$

where unit vectors $\boldsymbol{r}(t) \in \mathbb{R}^n$ and $\boldsymbol{y}(t) \in \mathbb{R}^n$ respectively correspond to the input and output signals, both of which are assumed accessible for all time $t$. Further, we assume the input vector $\boldsymbol{r}(t)$ to be a differentiable function of time with a bounded derivative.

From a practical standpoint, in three-dimensions, the measurement model Eq. (1) is typical for single input-output type unit vector measurement sensors such as star trackers, sun sensors and magnetometers[15]. More specific examples of star sensor modeling in attitude determination problems can be found in [16, 17, 18, 19].

The evolution of $C(t)$ in Eq. (1) is governed by the Poisson differential equation as described by

$$\dot{C}(t) = -S(\boldsymbol{\omega}(t))C(t) \tag{2}$$

where $\boldsymbol{\omega}(t) \in \mathbb{R}^m$ is any prescribed/measured bounded signal for $m = n(n-1)/2$, and $S(\cdot) : \mathbb{R}^m \to \mathbb{R}^{n \times n}$ is a skew-symmetric matrix function such that $S^T = -S$. It is a well known fact that if $C(0)$ is a proper orthogonal matrix, then $C(t)$ will remain a proper and orthogonal matrix for all $t$ whenever it evolves along Eq. (2)[20]. Now, the estimation objective is to find an appropriate adaptive algorithm that generates the proper orthogonal matrix $\hat{C}(t) \in \mathcal{R}^{n \times n}$ as an estimate for the "true" but unknown matrix $C(t)$ at each instant $t$. The output model based on $\hat{C}(t)$ can be established by

$$\hat{\boldsymbol{y}}(t) = \hat{C}(t)\boldsymbol{r}(t). \tag{3}$$

Discrete-time analogues of the aforementioned attitude estimation problem routinely arise within the field of spacecraft attitude determination. In such cases, $\boldsymbol{r}(t)$ and $\boldsymbol{y}(t)$ are respectively interpreted as the star catalog values (inertial) and the corresponding star tracker measurements. Estimation problems within this framework also occur in space applications of rendezvous and proximity operations that involve computation of the relative navigation solutions.

The attitude estimation error matrix $\tilde{C}(t)$ defined by

$$\tilde{C}(t) = \hat{C}(t) - C(t) \tag{4}$$

represents the element-by-element error between the estimate $\hat{C}(t)$ and the "true" attitude $C(t)$. Obvious from the definitions in Eq. (1) and Eq. (3) is the fact that the attitude error convergence, $\lim_{t\to\infty} \tilde{C}(t) = 0$, automatically implies the output estimation error, $\boldsymbol{e}(t) \doteq \hat{\boldsymbol{y}}(t) - \boldsymbol{y}(t) \to 0$ as $t \to \infty$. In the following developments, unless considered necessary, the time argument $t$ is omitted from various signals for the sake of notational simplicity.

## 2.2 Main Result

In this section, we present a new attitude estimation algorithm which preserves orthogonality of the estimated attitude matrix $\hat{C}(t)$.

**Theorem 1.** *If the attitude estimate $\hat{C}(t)$ is generated according to*

$$\dot{\hat{C}}(t) = -\left[S(\boldsymbol{\omega}) - \gamma(\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T)\right]\hat{C}(t); \quad \text{any real } \gamma > 0, \text{ any proper orthogonal } \hat{C}(0) \in \mathbb{R}^{n \times n} \tag{5}$$

*then, $\hat{C}(t)$ remains a proper orthogonal matrix for all $t > 0$ implying that attitude matrix estimation error $\tilde{C}(t)$ and the output estimation error $\boldsymbol{e}(t)$ remain bounded for all $t \geq 0$. In addition, the estimation process for $\hat{C}(t)$ is driven along an "attracting manifold" according to the following convergence condition*

$$\lim_{t\to\infty}\left(I_{n \times n} - C^T\hat{C}C^T\hat{C}\right)\boldsymbol{r}(t) = 0. \tag{6}$$

**Proof.** Just the same way that the Poisson differential equation Eq. (2) enforces othogonality on $C(t)$ for all $t$, the attitude estimate $\hat{C}(t)$ is also ensured to be proper and orthogonal by virtue of the fact that it is generated by another Poisson differential equation given in Eq. (5). This similarity can be made more explicit by recognizing that the additional term within Eq. (5) given by $(\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T)$ is also a skew-symmetric matrix. Since $\hat{C}(t)$ is guaranteed to be an orthogonal matrix for all $t \geq 0$, it follows that $\hat{C}(t)$ and $\tilde{C}(t)$ remain bounded for all $t \geq 0$. This is sufficient to demonstrate boundedness of the output estimation error $\boldsymbol{e}(t)$.

Next, in order to prove the convergence claim along the attracting manifold in Eq. (6), we consider the following Lyapunov candidate function

$$V(t) = \frac{1}{2}\text{tr}(\tilde{C}^T(t)\tilde{C}(t)) \tag{7}$$

where $\text{tr}(\cdot)$ is the matrix trace operator. Then, the time derivative of $V(t)$, evaluated along Eq. (2) and Eq. (5) is given by

$$\begin{aligned}
\dot{V}(t) &= \text{tr}\left(-\tilde{C}^T S(\boldsymbol{\omega})\tilde{C} + \gamma\tilde{C}^T(\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T)\hat{C}\right) \\
&= \gamma\text{tr}\left(\tilde{C}^T(\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T)\hat{C}\right) \\
&= \gamma\text{tr}\left((\hat{C}^T - C^T)(C\boldsymbol{r}\boldsymbol{r}^T - \hat{C}\boldsymbol{r}\boldsymbol{r}^T C^T\hat{C})\right) \\
&= \gamma\text{tr}\left(-\boldsymbol{r}\boldsymbol{r}^T + C^T\hat{C}C^T\hat{C}\boldsymbol{r}\boldsymbol{r}^T\right) \\
&= -\gamma\boldsymbol{r}^T(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r} \\
&= -\frac{\gamma}{2}\boldsymbol{r}^T(I_{n\times n} - C^T\hat{C}C^T\hat{C} - \hat{C}^T C\hat{C}^T C + I_{n\times n})\boldsymbol{r} \\
&= -\frac{\gamma}{2}\boldsymbol{r}^T(I_{n\times n} - C^T\hat{C}C^T\hat{C} - \hat{C}^T C\hat{C}^T C + (C^T\hat{C}C^T\hat{C})^T(C^T\hat{C}C^T\hat{C}))\boldsymbol{r} \\
&= -\frac{\gamma}{2}\boldsymbol{r}^T(I_{n\times n} - C^T\hat{C}C^T\hat{C})^T(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r} \\
&= -\frac{\gamma}{2}\|(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}\|^2
\end{aligned} \tag{8}$$

which demonstrates $\dot{V}(t) \leq 0$ and thereby uniform boundedness of $V(t)$. Since $V(t) \geq 0$ by definition from Eq. (7) and $\dot{V}(t) \leq 0$ from Eq. (8), we have existence of $V_\infty \doteq \lim_{t\to\infty} V(t)$. Further, from the fact that $\ddot{V}(t)$ is bounded (seen by differentiating both sides of Eq. (8)), using Barbalat's lemma, we conclude $\lim_{t\to\infty}(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}(t) = 0$. ■

Remark 1. In the case that the attitude estimate $\hat{C}(t)$ converges to the corresponding "true" attitude $C(t)$ (i.e., $\lim_{t\to\infty}\tilde{C}(t) = 0$), using the matrix orthogonality property, it is easy to recognize that $C^T(t)\hat{C}(t) \to I_{n\times n}$ as $t \to \infty$ which obviously satisfies the convergence condition of Theorem 1 given in Eq. (6). On the other hand, mere presence of the attracting manifold as governed by Eq. (6) does not in general guarantee regulation of attitude estimation error $\tilde{C}(t)$ to zero unless certain persistence of excitation (PE) conditions are additionally satisfied by the input signal $\boldsymbol{r}(t)$. These PE conditions will be further elaborated upon in the later sections of the paper.

### 2.3 Robustness to measurement noise

We next present robustness properties of the estimation algorithm from Theorem 1 to account for possible presence of bounded measurement noise. Introducing a bounded noise signal $\boldsymbol{v}(t)$ in Eq. (1) to reflect the presence of error in the measurement signal, we have

$$\boldsymbol{y}(t) = C(t)\boldsymbol{r}(t) + \boldsymbol{v}(t) \tag{9}$$

wherein we assume the noise signal satisfies $v_{\max} \doteq \sup_t\|\boldsymbol{v}(t)\|$. Even though not required for any of our further developments, from a practical standpoint, it is perfectly meaningful to assume that $v_{\max} \ll 1$. While retaining the same estimation rule for the attitude matrix $\hat{C}(t)$ as given by Eq. (5), we consider the following Lyapunov candidate function that enables demonstration of the robustness properties:

$$V_r(t) = \frac{1}{2}\mathrm{tr}\left[\tilde{C}^T\tilde{C}\right]. \tag{10}$$

Since presence of measurement noise $\boldsymbol{v}(t)$ in no way changes the fact that $\hat{C}(t)$, estimated according to Eq. (5), is a proper and orthogonal matrix for all $t \geq 0$, the function $V_r(t)$ is uniformly bounded. Further, using Eq. (3), Eq. (5), and Eq. (9), the time derivative of $V_r(t)$ can we written as as

$$\begin{aligned}
\dot{V}_r(t) &= \mathrm{tr}\left(\tilde{C}\dot{\hat{C}}\right) \\
&= \gamma\mathrm{tr}\left(\tilde{C}^T(\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T)\hat{C}\right) \\
&= -\gamma\boldsymbol{r}^T\left(I - C^T\hat{C}C^T\hat{C}\right)\boldsymbol{r} - \gamma\boldsymbol{v}^T C\left(I - C^T\hat{C}C^T\hat{C}\right)\boldsymbol{r} \\
&\leqslant -\frac{\gamma}{2}\|(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}\|\left[\|(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}\| - 2v_{\max}\right].
\end{aligned} \tag{11}$$

In the presence of measurement noise, the convergence condition of Theorem 1 along the attracting manifold $(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}(t) = 0$ is no longer preserved. However, it is also obvious from Eq. (11) that $\dot{V}_r(t) \leq 0$, whenever $\|(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}\| - 2v_{\max} \geq 0$ and therefore the size of the residual set governing $V_r(t)$ is ultimately dictated by the upper bound on the noise signal $\boldsymbol{v}(t)$. The foregoing analysis of the proposed attitude estimation algorithm provides confirmation that while all the computed signals remain bounded in the presence of measurement uncertainty, presence of large magnitude noise leads to increasing dilution of attitude estimation accuracy.

### 2.4 Estimation within the Certainty-Equivalence Framework

The main result of this paper derived in Theorem 1 will now be compared with an alternative estimation mechanism based on the traditional/conventional certainty-equivalence (CE) framework[21]. In order to be consistent with the standard assumptions of the CE methodology, we restrict the attitude matrix $C(t)$ in Eq. (1) to be a constant $C^*$ (i.e., $\boldsymbol{\omega}(t) = 0$ in Eq. (2)), and that there exists no measurement noise ($\boldsymbol{v}(t) = 0$). Important to mention here is the fact that recognition and utilization of prior information on the structure of unknown parameter $C^*$ leads us to nonlinear parameterization of the unknown elements in Eq. (1). On the other hand, if we ignore this prior information on orthogonality of $C^*$, then the unknown parameter appears linearly (affinely) in the governing input-output relationship, Eq. (1), thereby making feasible application of the CE-based adaptive estimation method. This convenient simplification comes at a heavy price: over-parameterization and/or permitting the parameter estimation (search) process for $\hat{C}(t)$ to potentially evolve outside the region where the true parameter $C^*$ lies - ultimately leading to the unacceptably slow/poor estimator performance. On the other hand, any attempt to explicitly utilize the a priori known parameter structure causes nonlinear parameterization which is not readily amenable to most existing CE-based formulations. For the case when $C(t) = C^*$, a simple CE-based formulation based on standard methods[21] may be obtained as given by

$$\dot{\hat{C}}(t) = -\gamma\boldsymbol{e}\boldsymbol{r}^T; \quad \text{any real } \gamma > 0, \text{ any } \hat{C}(0) \in \mathbb{R}^{n\times n} \tag{12}$$

where just as before, the output estimation error $\boldsymbol{e}(t)$ is defined by $\boldsymbol{e}(t) = \hat{\boldsymbol{y}}(t) - \boldsymbol{y}(t)$. Stability for the CE-based estimation algorithm given in Eq. (12) can be demonstrated by considering the following Lyapunov candidate function

$$V^{ce}(t) = \frac{1}{2}\mathrm{tr}\left(\tilde{C}^T\tilde{C}\right) \tag{13}$$

where $\tilde{C}(t)$ is defined in Eq. (4). Then, the time derivative of $V^{ce}(t)$ taken along solutions generated by Eq. (12) is given as

$$\begin{aligned}\dot{V}^{ce}(t) &= \mathrm{tr}\left(\tilde{C}^T\dot{\tilde{C}}\right) = -\gamma\mathrm{tr}\left(\tilde{C}^T\boldsymbol{e}\boldsymbol{r}^T\right) \\ &= -\gamma\mathrm{tr}\left(\tilde{C}^T\tilde{C}\boldsymbol{r}\boldsymbol{r}^T\right) = -\gamma\boldsymbol{r}^T\tilde{C}^T\tilde{C}\boldsymbol{r} \\ &= -\gamma\|\tilde{C}\boldsymbol{r}\|^2 = -\gamma\|\boldsymbol{e}\|^2 \ \leq 0 \end{aligned} \tag{14}$$

Since $V^{ce}(t) \geq 0$ and $\dot{V}^{ce}(t) \leq 0$, we have boundedness for the parameter estimation error $\tilde{C}(t)$ and therefore boundedness for the attitude estimate matrix $\hat{C}(t)$. Also, $V^{ce}_{\infty} \doteq \lim_{t\to\infty} V^{ce}(t)$ exists and is finite because $V^{ce}(t) \geq 0$ and $\dot{V}^{ce}(t) \leq 0$. Integrating both sides of Eq. (14), we can show that $V^{ce}_{\infty} - V_{ce}(0) = -\int_0^{\infty} \|\boldsymbol{e}(t)\|^2 \, dt$ which implies that $\boldsymbol{e} \in \mathcal{L}_2 \cap \mathcal{L}_{\infty}$. Furthermore, from the fact that $\dot{\boldsymbol{e}} = \dot{\tilde{C}}\boldsymbol{r} + \tilde{C}\dot{\boldsymbol{r}}$ whose terms are all bounded signals, we have $\dot{\boldsymbol{e}} \in \mathcal{L}_{\infty}$. Therefore, from Barbalat's lemma, we guarantee that the output estimation error $\boldsymbol{e}(t) \to 0$ as $t \to \infty$.

One very important point we should emphasize here is that $\lim_{t\to\infty} \boldsymbol{e}(t) = 0$ does not always imply $\lim_{t\to\infty} \tilde{C}(t) = 0$ unless the reference signal $\boldsymbol{r}(t)$ satisfies certain additional persistence of excitation (PE) conditions. For example, if $n = 2$ (two-dimensional case) and the reference input signal $\boldsymbol{r}(t) = \boldsymbol{r}^*$, i.e., a constant vector in $\mathbb{R}^2$, then $\boldsymbol{e} = \tilde{C}\boldsymbol{r}^* = 0$ can always be satisfied without $\tilde{C}$ equalling zero provided the vector $\boldsymbol{r}^*$ resides within the kernel (null space) of the matrix $\tilde{C}$.

A few important remarks are now in order to highlight the limitations of the CE-based attitude estimation scheme.

1. A proof of stability, boundedness, and convergence can be derived for the CE-based method (as outlined in the foregoing) only when the unknown attitude matrix remains constant with time. No such restriction exists on the result proposed under Theorem 1.
2. Under the CE-based estimation scheme, even when one selects the initial estimate $\hat{C}(0)$ to be proper and orthogonal, there is no assurance whatsoever that $\hat{C}(t)$ estimated through Eq. (12) remains orthogonal for $t > 0$. On the other hand, by the very fact that the attitude estimation under the proposed approach proceeds along Eq. (5) (Poisson differential equation), orthogonality of the estimate matrix $\hat{C}(t)$ is rigorously assured for all $t \geq 0$.
3. Further, in the next sections, we will show that when compared with CE-based attitude estimation, the main result of this paper needs weaker (less-restrictive) PE conditions on the reference input signal so as to ensure that the parameter estimation error converges to zero.

# 3. Attitude Estimation in Flatland

In this section, we specialize the main result for attitude matrix estimation to the two-dimensional (flatland) case, i.e., $n = 2$ in Eq. (1). While doing so, we will also be able to precisely characterize the persistence of excitation (PE) conditions that would regulate the attitude estimation error matrix $\tilde{C}(t)$ to zero. Our use of the term "flatland" is inspired and motivated to a great extent by the work of Shuster[22] that addressed attitude analysis in two-dimensions.

Note that for the flatland case, the true/unknown attitude matrix $C(t)$ and its estimate $\hat{C}(t)$ can be parameterized in terms of scalar "angle-like" variables respectively designated by $\theta(t)$ and $\hat{\theta}(t)$ in the following fashion

$$C(t) = e^{J\theta(t)} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \hat{C}(t) = e^{J\hat{\theta}(t)} = \begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} \\ \sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix} \tag{15}$$

where $J$ is the $2\times 2$ matrix generalization of $\sqrt{-1}$ and is given by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{16}$$

For the flatland case, the following result can be established as a direct consequence of Theorem 1.

**Corollary 1.** *Consider the input-output system described by Eq. (1), with $n = 2$. If the initial value of the attitude estimate $\hat{C}(0) = e^{J\hat{\theta}(0)}$ is such that $\hat{\theta}(0) \in \Theta_s$ where*

$$\Theta_s = \{\phi \in \mathbb{R} : \phi - \theta(0) \neq (2k+1)\pi\}, \quad k = 0, \pm 1, \pm 2, \ldots$$

*then the attitude estimate matrix $\hat{C}(t)$ generated through Eq. (5) exponentially converges to the unknown true value $C(t)$ for all non-zero (unit-vector) reference inputs $\boldsymbol{r}(t) \in \mathbb{R}^2$.*

**Proof.** We begin by specializing the Lyapunov candidate function $V(t)$ given in Eq. (7) for the two-dimensional (flatland) case as $V_F(t)$ which may readily be expressed in terms of the parameterization defined by Eq. (15) as follows:

$$\begin{aligned} V_F(t) &= \frac{1}{2}\text{tr}(\tilde{C}^T(t)\tilde{C}(t)) = \frac{1}{2}\text{tr}\left[(\hat{C}-C)^T(\hat{C}-C)\right] \\ &= \frac{1}{2}\text{tr}\left[2I_{2\times 2} - (\hat{C}^T C + C^T\hat{C})\right] = \frac{1}{2}\text{tr}\left[2I_{2\times 2} - e^{-J(\hat{\theta}-\theta)} - e^{J(\hat{\theta}-\theta)}\right] \\ &= 2\left[1 - \cos(\hat{\theta}-\theta)\right] = 4\sin^2\left(\frac{\hat{\theta}-\theta}{2}\right). \end{aligned} \tag{17}$$

It is clear from the last step of the preceding development that not only is $V_F(t)$ bounded for all $t \geq 0$ but in fact, we have $0 \leq V_F(t) \leq 4$ for all $t \geq 0$. Further, it can be seen that $V_F(t) = 4$, its maximum value, whenever $\hat{\theta}(t) - \theta(t) = (2k+1)\pi$ for all integers $k$. Thus, $V_F(0) < 4$ if $\hat{\theta}(0) \in \Theta_s$ is satisfied. On the other hand, $V_F(t) = 0$, its minimum value, whenever $\hat{\theta}(t) = \theta(t)$ (estimated variable $\hat{\theta}(t)$ exactly equals the true/unknown variable $\theta(t)$ value). Of course, more generally, $V_F(t) = 0$ whenever, $\hat{\theta}(t) - \theta(t) = 2k\pi$ for integers $k$. All the aforementioned interesting and important characteristics of the Lyapunov candidate function $V_F(t)$ for the flatland case are illustrated in Fig. 1.

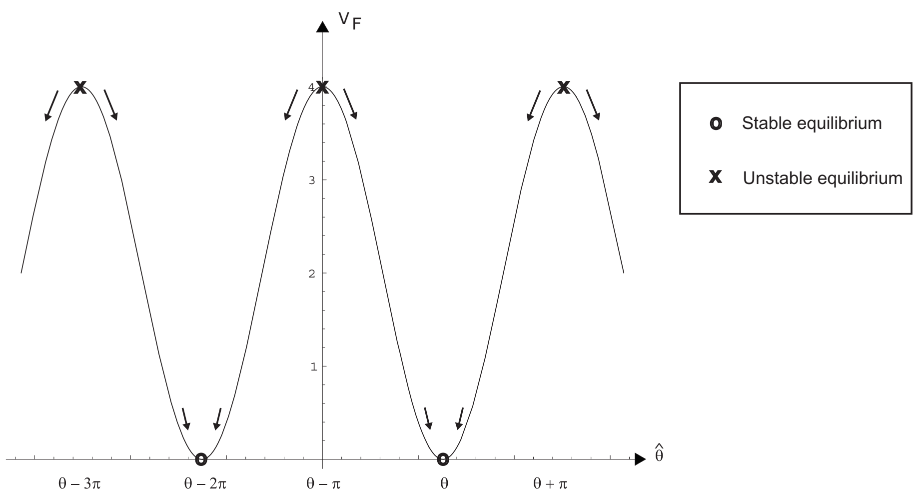



Figure 1: Plot showing variation of the bounded Lyapunov candidate function for flatland case $V_F(t)$ with respect to the estimated variable $\hat{\theta}(t)$.

Additional insights into the characterization of dynamics of $V_F(t)$ can be obtained by specializing function $\dot{V}(t)$ in Eq. (8) with $n = 2$ for the flatland case. Since the quantity $C^T\hat{C}C^T\hat{C}$ can be written as

$C^T\hat{C}C^T\hat{C} = e^{J(2\hat{\theta}-2\theta)}$ in the flatland case, making using of Eq. (17), we may derive a simplified expression for $\dot{V}_F(t)$ in the following fashion

$$\begin{aligned}
\dot{V}_F(t) &= -\gamma \boldsymbol{r}^T \left[I_{2\times 2} - e^{J(2\hat{\theta}-2\theta)}\right] \boldsymbol{r} \\
&= -\gamma \boldsymbol{r}^T \begin{bmatrix} 1-\cos(2\hat{\theta}-2\theta) & -\sin(2\hat{\theta}-2\theta) \\ \sin(2\hat{\theta}-2\theta) & 1-\cos(2\hat{\theta}-2\theta) \end{bmatrix} \boldsymbol{r} \\
&= -\gamma \left[1-\cos(2\hat{\theta}-2\theta)\right] \|\boldsymbol{r}\|^2 \\
&= -8\gamma \sin^2\left(\frac{\hat{\theta}-\theta}{2}\right)\cos^2\left(\frac{\hat{\theta}-\theta}{2}\right)\|\boldsymbol{r}\|^2 \\
&= -\frac{\gamma}{2} V_F(4-V_F)\|\boldsymbol{r}\|^2 \leq 0. 
\end{aligned} \tag{18}$$

As stated before, the reference input $\boldsymbol{r}(t)$ is an unit vector for all $t \geq 0$ and thus, we have

$$\dot{V}_F(t) = -\frac{\gamma}{2} V_F(t)\left[4 - V_F(t)\right] \tag{19}$$

which indicates the fact that the dynamics of $V_F(t)$ is governed by an infinite number of equilibrium points whenever $\dot{V}_F(t) = 0$ is satisfied along the $\hat{\theta}$-axis as can be seen in Fig. 1. However, those equilibrium points clearly segment into two distinct categories. One is an unstable branch in the sense that $\dot{V}_F(t) = 0$ while $V_F(t) = 4$, i.e., $\hat{\theta}(t) - \theta(t) = (2k+1)\pi$ for integer values of variable $k$. Thus, if $V_F(0) = 4$, then $V_F(t) = 4$ for all $t > 0$ irrespective of the reference input vector $\boldsymbol{r}(t)$. The other set of equilibrium points form a stable branch satisfying the conditions $\dot{V}_F(t) = 0$ and $V_F(t) = 0 \Longleftrightarrow \hat{\theta}(t) - \theta(t) = 2k\pi$ for all integers $k$. The equilibrium points corresponding to the condition $V_F(t) = 4$ are designated unstable because whenever $V_F(0) \neq 4$, we are guaranteed that $V_F(t) < 4$ for all $t > 0$ due to the fact that $\dot{V}_F(t) \leq 0$ for all $t \geq 0$. Clearly, if $\hat{\theta}(0) \in \Theta_s$, then $V_F(t) < 4$ for $t \geq 0$ is assured resulting in the fact that $\dot{V}_F(t) \to 0$ only when $V_F(t) \to 0$ (stable and attracting manifold).

The convergence properties of $V_F(t)$ starting from any initial value $V_F(0) \notin \{0, 4\}$ can be established by recognizing that the first order ordinary differential equation governing $V_F(t)$ in Eq. (19) admits an analytical solution as given by

$$V_F(t) = \frac{4ce^{-2\gamma t}}{1+ce^{-2\gamma t}} \quad \text{for all } t \geq 0, \quad \text{where } c = \frac{V_F(0)}{4-V_F(0)}. \tag{20}$$

Therefore it immediately follows that any initial condition $\hat{\theta}(0) \in \Theta_s$ implying $V_F(0) < 4$ leads to exponential convergence of $V_F(t)$ to zero. In terms of the attitude estimate matrix, this result means exponential convergence along $\hat{\theta}(t) - \theta(t) \to 2k\pi$ for all integers $k$, and accordingly, we conclude that $\tilde{C}(t) \to 0$ exponentially as $t \to \infty$. ■

In light of Corollary 1, the following observations are in order.

1. In flatland, whereas the $2 \times 2$ matrix $\hat{C}(t)$ has four elements that need to be estimated, the process of updating estimates of $\hat{C}(t)$ can be accomplished by updating a single scalar variable thereby eliminating any scope for overparameterization. More specifically, instead of directly estimating $\hat{C}(t)$ through the matrix differential equation of Eq. (5), one can efficiently generate the same $\hat{C}(t)$ matrix through the identity $\hat{C}(t) = e^{J\hat{\theta}(t)}$ given from Eq. (15). It is straightforward algebra to show that the "angle" variable $\hat{\theta}(t)$ needed for computation of the attitude estimate matrix $\hat{C}(t)$ is updated through the scalar differential equation

$$\dot{\hat{\theta}}(t) = \omega(t) + \gamma\left[y_2(t)\hat{y}_1(t) - y_1(t)\hat{y}_2(t)\right] \tag{21}$$

   where the scalar $\omega(t)$ can be physically interpreted as an "angular velocity" variable such that $\boldsymbol{\omega}(t) = \omega(t)\boldsymbol{e}_k$ in Eq. (2), and $\boldsymbol{e}_k$ denotes an unit vector normal to the flatland.

2. Exponential convergence of $\tilde{C}(t)$ to zero is guaranteed regardless of $\boldsymbol{r}(t)$ (non-zero) values so long as $\hat{\theta}(0) \in \Theta_s$. This remarkable feature holds true without requiring the unknown matrix $C(t)$ to be a constant, i.e., $\boldsymbol{\omega}(t) \neq 0$ in Eq. (2). Further, the same exponential convergence condition holds even when $\boldsymbol{r}(t) = \boldsymbol{r}^*$ (any non-zero constant vector) and thus every non-zero reference input vector is persistently exciting in the flatland case. As already mentioned in the previous section, similar convergence assurances are impossible from the conventional CE-based estimation framework.

3. If $\hat{\theta}(0) \notin \Theta_s$, then $V_F(t) = V_F(0) = 4$ for all $t > 0$ (the unstable equilibrium branch) and obviously, there is no way to regulate the estimation error $\tilde{C}(t)$ to zero irrespective of the reference input $\boldsymbol{r}(t)$. However, from a practical standpoint, even the smallest of perturbations and/or numerical drift that causes $V_F(t)$ to deviate from $V_F(0)$ at some $t = t^* > 0$ would ensure that $\hat{\theta}(t) \in \Theta_s$ for all $t \geq t^*$ (the stable and attracting manifold) and consequently we are again assured of $\tilde{C}(t)$ exponentially converging to zero on $t \geq t^*$.

4. In the presence of bounded measurement noise $\boldsymbol{v}(t)$, the robustness result for the general $n$-dimensional case from the previous section can be specialized for the flatland case by rewriting Eq. (11) as follows:

$$\begin{aligned} \dot{V}_r(t) &= -2\gamma \sin^2(\hat{\theta} - \theta)\|\boldsymbol{r}\|^2 - \gamma \left[\cos\theta - \cos(2\hat{\theta} - \theta)\right] \boldsymbol{v}^T \boldsymbol{r} \\ &\leq -2\gamma \sin^2(\hat{\theta} - \theta)\|\boldsymbol{r}\|^2 + 2\gamma \sin(\hat{\theta} - \theta)\boldsymbol{v}^T \boldsymbol{r}. \end{aligned} \tag{22}$$

Recalling the reference input $\boldsymbol{r}(t)$ to be an unit vector, the previous inequality may further be simplified and arranged as

$$\dot{V}_r(t) \leq -2\gamma\, |\sin(\hat{\theta} - \theta)|\, \Big[|\sin(\hat{\theta} - \theta)| - v_{\max}\Big] \tag{23}$$

It is clear from here that $\dot{V}_r(t) \leq 0$ whenever $|\sin(\hat{\theta} - \theta)| - v_{\max} \geq 0$. Further, Eq. (23) can also be adopted for the case of relatively low magnitude measurement noise (i.e., $v_{\max}$ is small enough for the small angle approximation $\sin(v_{\max}) \approx v_{\max}$ to hold), so that as $t \to \infty$, the inequality $|\hat{\theta}(t) - \theta(t)| \leq v_{\max}$ serves as an useful approximate upper bound on the estimation error. Clearly, if $v_{\max} = 0$ is applied in this approximation result (the ideal no-noise case), we immediately recover the (exponential) convergence property $\lim_{t\to\infty} \hat{\theta}(t) - \theta(t) = 0$ which was established earlier in this section.

# 4. Attitude Estimation in Three-Dimensions

Although the input-output model in Eq. (1) is for a single measurement system, the results from Theorem 1 can be readily extended to the systems having multiple measurement devices. Let $M$ denote the total number of measurement devices, i.e.,

$$\boldsymbol{y}_k(t) = C(t)\boldsymbol{r}_k(t), \quad k = 1, 2, \ldots, M \tag{24}$$

Then, the update law from Eq. (5) needs to be modified to account for the presence of more than one input-output pair and is now given by

$$\dot{\hat{C}}(t) = -\left[S(\boldsymbol{\omega}) - \gamma \sum_{k=1}^{M} (\boldsymbol{y}_k \hat{\boldsymbol{y}}_k^T - \hat{\boldsymbol{y}}_k \boldsymbol{y}_k^T)\right] \hat{C}(t); \quad \text{any real } \gamma > 0, \text{ any proper orthogonal } \hat{C}(0) \in \mathbb{R}^{n \times n} \tag{25}$$

wherein the $k^{\text{th}}$ estimated output $\hat{\boldsymbol{y}}_k(t)$ is defined through

$$\hat{\boldsymbol{y}}_k(t) = \hat{C}(t)\boldsymbol{r}_k(t), \quad k = 1, 2, \ldots, M \tag{26}$$

From the same definition of Lyapunov candidate Eq. (7) with a subscript $M$, we can derive the following time derivative of $V_M(t)$:

$$\begin{aligned}\dot{V}_M(t) &= \mathrm{tr}\left(-\tilde{C}^T S(\boldsymbol{\omega})\tilde{C} + \gamma \sum_{k=1}^{M} \tilde{C}^T (\boldsymbol{y}_k \hat{\boldsymbol{y}}_k^T - \hat{\boldsymbol{y}}_k \boldsymbol{y}_k^T)\hat{C}\right) \\ &= \gamma \mathrm{tr}\left(\sum_{k=1}^{M}(-\boldsymbol{r}_k \boldsymbol{r}_k^T + C^T\hat{C}C^T\hat{C}\boldsymbol{r}_k\boldsymbol{r}_k^T)\right) \\ &= -\gamma \sum_{k=1}^{M}\left(\boldsymbol{r}_k^T(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}_k\right) \\ &= -\frac{\gamma}{2}\sum_{k=1}^{M}\|(I_{n\times n} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}_k\|^2 \end{aligned} \tag{27}$$

which identically covers Eq. (8) when $M = 1$. Stability proof remains the same as Theorem 1 except for the summation. One thing should be noted in Eq. (27) is that we can obtain the same effect from a single measurement device by tuning $\gamma$ of Eq. (5) instead of adopting $M$ measurement devices. Given that all technical details remain unaltered with single or multiple measurements, to keep the notation simple, we retain the single measurement model while discussing all further implications of our proposed attitude estimation algorithm for three-dimensions.

We adopt the four-dimensional unit-norm constrained quaternion vector representation for a singularity-free parameterization of the attitude matrix while specializing the results of Theorem 1 to the case of three-dimensions ($n = 3$). Accordingly, we assume that the true attitude matrix $C(t)$ is represented by the quaternion vector $\boldsymbol{q}(t) = [q_o(t), \boldsymbol{q}_v(t)]^T$ where the subscripts '$o$' and '$v$' respectively designate the scalar and vector parts of the quaternion representation. This quaternion parameterization is mathematically realized through the relationship

$$C(t) = I_{3\times3} - 2q_o(t)S(\boldsymbol{q}_v(t)) + 2S^2(\boldsymbol{q}_v(t)) \tag{28}$$

where the $3\times3$ skew-symmetric matrix operator $S(\cdot)$ designates the vector cross product operation such that $S(\boldsymbol{a})\boldsymbol{b} = \boldsymbol{a}\times\boldsymbol{b}$ for all three-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Similarly, the estimate matrix $\hat{C}(t)$ is represented through the quaternion parameterization $\hat{\boldsymbol{q}}(t) = [\hat{q}_o(t), \hat{\boldsymbol{q}}_v(t)]^T$ so that we have

$$\hat{C}(t) = I_{3\times3} - 2\hat{q}_o(t)S(\hat{\boldsymbol{q}}_v(t)) + 2S^2(\hat{\boldsymbol{q}}_v(t)) \tag{29}$$

Further, if we designate the quaternion $\boldsymbol{z}(t) = [z_o(t), \boldsymbol{z}_v(t)]^T$ to parameterize the proper orthogonal matrix $C^T(t)\hat{C}(t)$, it follows that

$$C^T(t)\hat{C}(t) = I_{3\times3} - 2z_o(t)S(\boldsymbol{z}_v(t)) + 2S^2(\boldsymbol{z}_v(t)) \tag{30}$$

where, by virtue of the quaternion multiplication property[23], the following identity holds

$$z_o(t) = \boldsymbol{q}^T(t)\hat{\boldsymbol{q}}(t). \tag{31}$$

In other words, if vectors $\boldsymbol{q}$ and $\hat{\boldsymbol{q}}$ are aligned in the same direction (i.e., if $\boldsymbol{q} = \pm\hat{\boldsymbol{q}}$ or in other words, $C = \hat{C}$), we have $z_o = \pm1$ which from the unit vector constraint on the quaternion implies that $\|\boldsymbol{z}_v\| = 0$. Recalling the definition of the attitude estimation error matrix $\tilde{C}(t)$ in Eq. (4), it is obvious that $\tilde{C} = C(C^T\hat{C} - I_{3\times3})$, and accordingly, we have $\tilde{C}(t) = 0$ whenever $C^T(t)\hat{C}(t) = I_{3\times3}$. Thus, the vector $\boldsymbol{z}(t)$ has the interpretation of the error quaternion such that $z_o(t) = \pm1 \iff \boldsymbol{z}_v(t) = 0$ if and only if matrix $C^T(t)\hat{C}(t) = I_{3\times3} \iff \tilde{C}(t) = 0$.

Note that the cascade matrix $C^T\hat{C}C^T\hat{C}$ is encountered in the convergence result of Theorem 1. Accordingly, we adopt yet another quaternion representation for the proper orthogonal matrix $C^T\hat{C}C^T\hat{C}$ given by $\boldsymbol{w}(t) = [w_o(t), \boldsymbol{w}_v(t)]^T$ such that

$$C^T(t)\hat{C}(t)C^T(t)\hat{C}(t) = I_{3\times3} - 2w_o(t)S(\boldsymbol{w}_v(t)) + 2S^2(\boldsymbol{w}_v(t)) \tag{32}$$

Through the quaternion multiplication property, it is easy to establish the following identities

$$w_o = 2z_o^2 - 1, \quad \boldsymbol{w}_v = 2z_o\boldsymbol{z}_v \tag{33}$$

We are now ready to state the following result applicable to the three-dimensional case.

**Corollary 2.** *For the given input-output system of Eq. (1) in the three-dimensional case, suppose the true/unknown attitude matrix $C(t)$ evolving according to Eq. (2) is parameterized by the quaternion vector $\boldsymbol{q}(t)$ through Eq. (28). If the attitude estimate matrix $\hat{C}(t)$ is parameterized by the unit quaternion $\hat{\boldsymbol{q}}(t)$ according to Eq. (29) and is updated according to Eq. (5) subject to the condition that $\hat{\boldsymbol{q}}(0) \notin \Psi_u$ where*

$$\Psi_u = \left\{ \boldsymbol{\eta} \in \mathbb{R}^4 : \|\boldsymbol{\eta}\| = 1; \boldsymbol{\eta}^T \boldsymbol{q}(0) = 0 \right\}$$

*then for all non-zero (unit-vector) reference inputs $\boldsymbol{r}(t) \in \mathbb{R}^3$ the following convergence condition holds asymptotically*

$$\lim_{t\to\infty} \|\boldsymbol{r}(t) \times \boldsymbol{z}_v(t)\| = 0 \tag{34}$$

*where $\boldsymbol{z}(t) = [z_o(t), \boldsymbol{z}_v(t)]^T$ is a quaternion representation for the cascaded proper orthogonal matrix $C^T(t)\hat{C}(t)$.*

**Proof.** Consider the same Lyapunov candidate function from Eq. (7) specialized to the three-dimensional case ($n = 3$) as follows

$$V_o(t) = \frac{1}{2}\mathrm{tr}\left(\tilde{C}^T\tilde{C}\right) = \frac{1}{2}\mathrm{tr}\left(2I_{3\times3} - (\hat{C}^T C + C^T\hat{C})\right) \tag{35}$$

Making use of the quaternion parameterization $\boldsymbol{z}(t)$ for the proper orthogonal matrix $C^T(t)\hat{C}(t)$ given in Eq. (30), we may rewrite $V_o(t)$ in the following manner:

$$\begin{aligned} V_o(t) &= \frac{1}{2}\mathrm{tr}\left[-2S^2(\boldsymbol{z}_v) - 2S^2(\boldsymbol{z}_v)\right] = -2\mathrm{tr}\left[S^2(\boldsymbol{z}_v)\right] \\ &= -2\mathrm{tr}\left[\begin{pmatrix} -z_3^2 - z_2^2 & z_1z_2 & z_1z_3 \\ z_1z_2 & -z_1^2 - z_3^2 & z_2z_3 \\ z_1z_3 & z_2z_3 & -z_2^2 - z_1^2 \end{pmatrix}\right] \\ &= 4(z_1^2 + z_2^2 + z_3^2) = 4\boldsymbol{z}_v^T\boldsymbol{z}_v \\ &= 4(1 - z_0^2) \end{aligned} \tag{36}$$

where $\boldsymbol{z}_v = [z_1, z_2, z_3]^T$. Now, it is clear that $V_o(t)$ is an uniformly bounded function for all $t \geq 0$. More specifically, $0 \leq V_o(t) \leq 4$ and there exists a very interesting geometric interpretation for $V_o(t)$ in Eq. (36) which can be explained through the quantities $\hat{\boldsymbol{q}}$ and $\boldsymbol{q}$ using the quaternion multiplication identity of Eq. (31). If $\hat{\boldsymbol{q}}$ is on the hyperplane normal to $\boldsymbol{q}$ (i.e., $z_0 = \boldsymbol{q}^T\hat{\boldsymbol{q}} = 0$), then $V_o = 4$, its maximum value. On the other hand, if $\hat{\boldsymbol{q}}$ is aligned along $\boldsymbol{q}$ (i.e., $z_0 = 1$ or $-1$), then $V_o = 0$, its minimum value. Physically, $z_o = 0$ corresponds to an error in the Euler principal rotation angle (for the matrix $C^T\hat{C}$) given by $\pm\pi$ and therefore, the properties of $V_o(t)$ in the three-dimensional case are very much analogous to the function $V_F(t)$ of the flatland case.

Starting from Eq. (8), and making use of the quaternion representation given in Eq. (32) for the matrix $C^T\hat{C}C^T\hat{C}$ together with the identities listed in Eq. (33), the time-derivative of function $V_o(t)$ can be written as follows

$$\begin{aligned} \dot{V}_o(t) &= -\frac{\gamma}{2}\|(I_{3\times3} - C^T\hat{C}C^T\hat{C})\boldsymbol{r}\|^2 \\ &= -\gamma\boldsymbol{r}^T\left[(1 - 2z_0^2)(2z_0)S(\boldsymbol{z}_v) - 8z_0^2S^2(\boldsymbol{z}_v)\right]\boldsymbol{r} \\ &= -8\gamma z_0^2\boldsymbol{r}^TS^T(\boldsymbol{z}_v)S(\boldsymbol{z}_v)\boldsymbol{r} \\ &= -8\gamma z_0^2\|\boldsymbol{z}_v \times \boldsymbol{r}\|^2 \quad \leq 0. \end{aligned} \tag{37}$$

From the foregoing analysis, it is clear that the dynamics of $V_o(t)$ have an equilibrium manifold corresponding to $z_o(0) = 0$ which corresponds to the set of all possible initial conditions $\hat{\boldsymbol{q}}(0) \in \Psi_u$. In this case, $V_o(0) = 4$ and $\dot{V}_o(t) = 0$ for all $t \geq 0$ resulting in the fact that $V_o(t)$ remains fixed at its initial value. Instability of this equilibrium manifold $\Psi_u$ can be demonstrated by the argument that if $z_o(0) \doteq \xi \neq 0$, with $\xi$ being arbitrarily small, then due to $\dot{V}_o(t) \leq 0$ from Eq. (37), we are guaranteed that $|z_o(t)| \geq |\xi|$ for all $t > 0$ and accordingly, there is no way for convergence $z_o(t) \to 0$ to happen as $t \to \infty$. We geometrically depict this interpretation of the unstable manifold in Fig. 2.

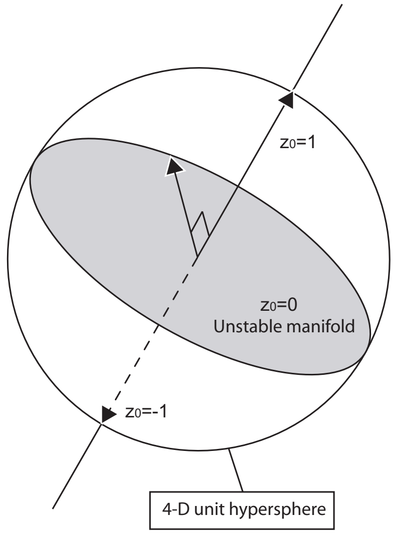



Figure 2: Illustration of the error quaternion vector $\boldsymbol{z} = [z_o, \boldsymbol{z}_v]^T$ representing the proper orthogonal matrix $C^T\hat{C}$. The hyperplane $z_o = 0$ represents an unstable equilibrium manifold.

On the other hand, assuming $\hat{\boldsymbol{q}}(0) \notin \Psi_u$, we know that $V_o(0) < 4$ thus precluding the possibility $\lim_{t\to\infty} z_o(t) = 0$. Using the facts $V_o(t) \geq 0$ and $\dot{V}_o(t) \leq 0$, we are assured the limit as $t \to \infty$ of $V_o(t)$ exists and is finite. As a consequence, the integral $\int_0^t \dot{V}_o(t)dt$ also exists as $t \to \infty$. Further, from boundedness of $\ddot{V}_o(t)$ (seen by differentiating both sides of Eq. (37) and recognizing that each term therein is bounded), we have uniform continuity of $\dot{V}_o(t)$. Using Barbalat's lemma, we are therefore led to conclude that $\lim_{t\to\infty} \dot{V}_o(t) = 0$. From Eq. (37), since we already ruled out the possibility of $\lim_{t\to\infty} z_o(t) = 0$ from happening, the only other remaining way for $\lim_{t\to\infty} \dot{V}_o(t) = 0$ to be satisfied is when $\lim_{t\to\infty} \|\boldsymbol{r}(t) \times \boldsymbol{z}_v(t)\| = 0$ thereby completing the proof. ■

For the case of attitude matrix estimation in three-dimensions, the following observations are now in order.

1. An important consequence of Corollary 2 is the fact that whenever $\hat{\boldsymbol{q}}(0) \notin \Psi_u$, not only are we assured of convergence $\lim_{t\to\infty} \|\boldsymbol{r}(t) \times \boldsymbol{z}_v(t)\| = 0$ but in fact, the output estimation error $\boldsymbol{e}(t)$ also converges to zero as $t \to \infty$. This result can be easily established as follows: first, from Eq. (30) we note that $\boldsymbol{z}_v \times \boldsymbol{r} = S(\boldsymbol{z}_v)\boldsymbol{r} = 0$ implies $C^T\hat{C}\boldsymbol{r} = \boldsymbol{r}$; subsequently, starting with the definition of the output estimation error $\boldsymbol{e}(t)$, in the limit $t \to \infty$, we obtain

$$\boldsymbol{e}(t) = \hat{\boldsymbol{y}}(t) - \boldsymbol{y}(t) = \tilde{C}(t)\boldsymbol{r}(t) = C(t)[C^T(t)\hat{C}(t) - I_{3\times 3}]\boldsymbol{r}(t) = 0$$

   which proves the stated assertion.

2. In three-dimensions, the attitude estimate $\hat{C}(t)$ is $3 \times 3$ matrix and hence has nine entries that need to be updated if we are to adopt the matrix differential equation described by Eq. (2). However, a computationally efficient method for estimating the $\hat{C}(t)$ matrix would be by updating the four-dimensional quaternion vector $\hat{\boldsymbol{q}}(t)$ satisfying the parameterization of Eq. (29). Such an update differential equation for $\hat{\boldsymbol{q}}(t)$ can indeed be derived. To do so, we first recognize that for three-dimensional vectors $\boldsymbol{y}(t)$ and $\hat{\boldsymbol{y}}(t)$, the skew-symmetric matrix $\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T$ listed in Eq. (2) can be expressed in terms of the vector cross-product as follows

$$\gamma(\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T) = -S(\gamma\boldsymbol{y} \times \hat{\boldsymbol{y}}) \tag{38}$$

   Accordingly, an update law for the unit-quaternion $\hat{\boldsymbol{q}}(t) = [q_o(t), \hat{\boldsymbol{q}}_v(t)]^T$ may be expressed by

$$\dot{\hat{q}}_o(t) = -\frac{1}{2}\hat{\boldsymbol{q}}_v^T(\boldsymbol{\omega} + \gamma\boldsymbol{y} \times \hat{\boldsymbol{y}}); \quad \dot{\hat{\boldsymbol{q}}}_v(t) = \frac{1}{2}[\hat{q}_o I_{3\times 3} + S(\hat{\boldsymbol{q}}_v)](\boldsymbol{\omega} + \gamma\boldsymbol{y} \times \hat{\boldsymbol{y}}) \tag{39}$$

3. The error quaternion vector $\boldsymbol{z}(t) = [z_o(t), \boldsymbol{z}_v(t)]^T$ parameterizing the proper orthogonal matrix $C^T(t)\hat{C}(t)$ according to Eq. (30) has time-evolution described by

$$\dot{z}_o(t) = -\frac{1}{2}\boldsymbol{z}_v^T C^T(t)(\gamma\boldsymbol{y} \times \hat{\boldsymbol{y}}); \quad \dot{\boldsymbol{z}}_v(t) = \frac{1}{2}[z_o I_{3\times 3} + S(\boldsymbol{z}_v)]C^T(t)(\gamma\boldsymbol{y} \times \hat{\boldsymbol{y}}) \tag{40}$$

   Using the rotational invariance property of the cross product,

$$C^T(t)(\gamma\boldsymbol{y} \times \hat{\boldsymbol{y}}) = \gamma C^T(t)\boldsymbol{y} \times C^T(t)\hat{\boldsymbol{y}} = \gamma\boldsymbol{r} \times C^T(t)\hat{C}(t)\boldsymbol{r}$$

   which can be substituted in Eq. (40) to establish the following

$$\dot{z}_o(t) = -\frac{\gamma}{2}\boldsymbol{z}_v^T[\boldsymbol{r} \times C^T(t)\hat{C}(t)\boldsymbol{r}]; \quad \dot{\boldsymbol{z}}_v(t) = \frac{\gamma}{2}[z_o I_{3\times 3} + S(\boldsymbol{z}_v)][\boldsymbol{r} \times C^T(t)\hat{C}(t)\boldsymbol{r}] \tag{41}$$

4. From corollary 2, it is possible to show that when $\hat{\boldsymbol{q}}(0) \notin \Psi_u$ together with $\dot{\boldsymbol{r}}(t) \neq 0$ (i.e., the reference input not a constant vector), then the attitude estimation error $\tilde{C}(t)$ asymptotically converges to zero. In order to prove this statement, we start from corollary 2 where we already proved that $\lim_{t\to\infty}[\boldsymbol{z}_v(t) \times \boldsymbol{r}(t)] = 0$. This result can be applied in Eq. (30) to infer that $C^T(t)\hat{C}(t)\boldsymbol{r}(t) = \boldsymbol{r}(t)$ as $t \to \infty$ which may further be substituted in Eq. (41) leading us to $\lim_{t\to\infty}\dot{\boldsymbol{z}}_v(t) = 0$. Further, from uniform continuity of $\boldsymbol{z}_v(t) \times \boldsymbol{r}(t)$, we have

$$\lim_{t\to\infty}[\boldsymbol{z}_v(t) \times \boldsymbol{r}(t)] = 0 \Longrightarrow \lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{dt}}[\boldsymbol{z}_v(t) \times \boldsymbol{r}(t)] = 0$$

   which implies that $\lim_{t\to\infty}[\dot{\boldsymbol{z}}_v(t) \times \boldsymbol{r}(t) + \boldsymbol{z}_v(t) \times \dot{\boldsymbol{r}}(t)] = 0$ or $\lim_{t\to\infty}[\boldsymbol{z}_v(t) \times \dot{\boldsymbol{r}}(t)] = 0$. So far, we have proved that as $t \to \infty$, we have quantities $\boldsymbol{z}_v(t) \times \dot{\boldsymbol{r}}(t) \to 0$ and $\boldsymbol{z}_v(t) \times \boldsymbol{r}(t) \to 0$, i.e., the vector $\boldsymbol{z}_v(t)$ is simultaneously parallel to both $\boldsymbol{r}(t)$ and $\dot{\boldsymbol{r}}(t)$. This can be possible only if $\boldsymbol{z}_v(t) = 0$ since every unit vector $\boldsymbol{r}(t)$ satisfies $\boldsymbol{r}^T(t)\dot{\boldsymbol{r}}(t) = 0$, and therefore vectors $\boldsymbol{r}(t)$ and $\dot{\boldsymbol{r}}(t)$ remain normal to one other for all $t$. Now that we have $\lim_{t\to\infty}\boldsymbol{z}_v(t) = 0$, from Eq. (30), it is possible to conclude that $C^T(t)\hat{C}(t) \to I_{3\times 3}$ as $t \to \infty$ and accordingly, the asymptotic convergence result $\lim_{t\to\infty}\tilde{C}(t) = 0$ for the attitude estimation error matrix.

5. We note that our proposed attitude estimation algorithm is developed under the assumption that the angular velocity vector $\boldsymbol{\omega}(t)$ is perfectly measured/determined. This assumption is easily violated in practical applications due to the fact that gyros that measure angular velocity exhibit drift over time. Typically, such gyro drift is slow and small (at least for high-grade gyros) and hence the bias in angular velocity measurements may be considered to remain constant. When angular velocity biases are included, the stability of the proposed estimator may be presented in a robustness context. Suppose a small constant bias $\boldsymbol{b}$ exists in the measurement of the angular velocity $\boldsymbol{\omega}(t)$, i.e.,

$$\tilde{\boldsymbol{\omega}}(t) = \boldsymbol{\omega}(t) + \boldsymbol{b}$$

   and the attitude estimate update law from Eq. (5) is accordingly modified as follows:

$$\dot{\hat{C}}(t) = -\left[S(\tilde{\boldsymbol{\omega}}) - \gamma(\boldsymbol{y}\hat{\boldsymbol{y}}^T - \hat{\boldsymbol{y}}\boldsymbol{y}^T)\right]\hat{C}(t); \quad \text{any real } \gamma > 0, \text{ any proper orthogonal } \hat{C}(0) \in \mathbb{R}^{n\times n} \tag{42}$$

   Then, the dynamics of the vector part of the error quaternion vector $\boldsymbol{z}_v(t)$ is modified from Eq. (41) and can be determined as follows

$$\dot{\boldsymbol{z}}_v(t) = \frac{\gamma}{2}[z_o I_{3\times 3} + S(\boldsymbol{z}_v)][\boldsymbol{r} \times C^T(t)\hat{C}(t)\boldsymbol{r}] + \frac{1}{2}[z_o I_{3\times 3} + S(\boldsymbol{z}_v)]C^T(t)\boldsymbol{b} \tag{43}$$

   When compared to the no bias case of Eq. (41), we note that presence of gyro bias introduces the bounded perturbation term seen from the last term of Eq. (43). This perturbation vanishes when $\boldsymbol{b} = 0$ but is otherwise non-vanishing when $\boldsymbol{z}_v(t) \to 0$.

   From a dynamic stability standpoint, obviously, we are now looking at an asymptotically stable system Eq. (41) being perturbed by a bounded perturbation. Linearization of the unperturbed system Eq. (41)

and evaluating about the equilibrium $\boldsymbol{z}_v = 0$ results in a time-varying Jacobian matrix $(\boldsymbol{r}\boldsymbol{r}^T - I_{3\times3})$ whose eigenvalues are evaluated as $\{-1, -1, 0\}$. The fact that the eigenvalues don't all have strictly negative real parts prevents us from using any of the stability/robustness results available through the linearized approximations. On the other hand, the unit norm constraint on $\boldsymbol{z}_v(t)$ obviously holds even in the presence of the bias error, and hence boundedness of trajectories for Eq. 43 in the presence of perturbations is not an issue. However, an interesting theoretical question is whether a meaningful upper bound can be derived on the difference between the solutions of the unperturbed system of Eq. (41) and the perturbed system in Eq. (43). It is reasonable to hypothesize that small bias errors (small perturbations) would maintain solutions of the perturbed system to remain "close" to those of the asymptotically stable unperturbed system. However, such conclusions are meaningful and may only be drawn on compact time intervals since difference between the unperturbed and perturbed solutions is governed through exponential terms involving the Lipschitz constants of the unperturbed system which grow to infinity as $t \to \infty$. A great technical obstacle for drawing bounds valid for all time $t$ is the fact that we only have proof of asymptotic stability for the $\boldsymbol{z}_v(t)$ states in the unperturbed case. In other words, a lack of proof for exponential stability for the nonlinear unperturbed system prevents us from making any authoritative comments on how far the trajectories of the perturbed system deviate from those of the unperturbed (no gyro bias case) nominal system. Of course, the fact that we cannot show that the difference between the unperturbed and perturbed trajectories remains small (at least when the bias error is small) doesn't mean it is not true[§].

6. It is possible to derive further insights into the convergence properties of the attitude estimation algorithm for the case of constant reference input, i.e., $\boldsymbol{r}(t) = \boldsymbol{r}_*$ for all $t \geq 0$. To enable this analysis, using Eq. (40), we derive the projection of $\dot{\boldsymbol{z}}_v(t)$ along $\boldsymbol{r}_*$ as given by

$$\begin{aligned} \boldsymbol{r}_*^T \dot{\boldsymbol{z}}_v(t) &= \frac{\gamma}{2}\boldsymbol{r}_*^T \left[z_o I_{3\times3} + S(\boldsymbol{z}_v)\right] C^T(t)(\boldsymbol{y} \times \hat{\boldsymbol{y}}) \\ &= \gamma \boldsymbol{r}_*^T \left[z_o I_{3\times3} + S(\boldsymbol{z}_v)\right] \left[z_o(\boldsymbol{z}_v \times \boldsymbol{r}_*) \times \boldsymbol{r}_* - (\boldsymbol{r}_*^T \boldsymbol{z}_v)(\boldsymbol{z}_v \times \boldsymbol{r}_*)\right] \\ &= \gamma(\boldsymbol{r}_*^T \boldsymbol{z}_v)\|\boldsymbol{z}_v \times \boldsymbol{r}_*\|^2 \end{aligned} \tag{44}$$

The preceding expression for $\boldsymbol{r}_*^T \dot{\boldsymbol{z}}_v(t)$ has two important consequences. The first is that when the component of $\boldsymbol{z}_v(t)$ along $\boldsymbol{r}_*$ is zero, then $\boldsymbol{z}_v(t)$ will always remain bounded to a plane that is normal to the $\boldsymbol{r}_*$ vector. This implies that the vector part of the quaternion estimation error vector $\boldsymbol{z}_v(t) \to 0$ as $t \to \infty$ whenever $\hat{\boldsymbol{q}}(0) \notin \Psi_u$, i.e., $z_o(0) \neq 0$, because of the fact that $\lim_{t\to\infty}[\boldsymbol{z}_v(t) \times \boldsymbol{r}_*] = 0$ as provided by corollary 2. The other consequence of Eq.(44) is that any non-zero component of $\boldsymbol{z}_v(t)$ along the direction of $\boldsymbol{r}_*$ has the same sign as that of the component of $\dot{\boldsymbol{z}}_v(t)$ along $\boldsymbol{r}_*$. Therefore, the absolute value of $\boldsymbol{r}_*^T \boldsymbol{z}_v(t)$ increases monotonically so that the vector part of the error quaternion $\boldsymbol{z}_v(t)$ never converges to zero.

The foregoing discussion can be formalized through an elegant geometric interpretation about an unit sphere in three-dimensions. Note that the vector part of the quaternion estimation error $\boldsymbol{z}_v(t)$ evolves on or inside a unit three-dimensional sphere. Now, consider the constant reference input vector $\boldsymbol{r}_*$ as the axis of this sphere so that its intercepts with the sphere are respectively the North and South poles. The plane normal to the $\boldsymbol{r}_*$ direction may be designated the equatorial plane. With these definitions, now, depending on various possibilities for the initial conditions on $\boldsymbol{z}_v(0)$ and $z_o(0)$, we have the following convergence conditions:

(a) If $\boldsymbol{z}_v(0)$ is on the equator, i.e., $z_o(0) = 0$ and $\boldsymbol{z}_v(0)^T\boldsymbol{r}_* = 0$, then $\boldsymbol{z}_v(t)$ will always remain on the equator.

(b) If $\boldsymbol{z}_v(0)$ is on the equatorial plane but not on the equator, i.e., $z_o(0) \neq 0$ and $\boldsymbol{z}_v(0)^T\boldsymbol{r}_* = 0$, the vector part of the quaternion estimation error $\boldsymbol{z}_v(t)$ asymptotically converges to zero (the center of the sphere). Accordingly, the attitude estimation matrix $\tilde{C}(t)$ also converges to zero.

(c) If $\boldsymbol{z}_v(0)$ is inside the sphere but not on the equatorial plane, then $\boldsymbol{z}_v(t)$ will converge to a point on the axis towards the closest pole.

[§]We acknowledge the insight provided to us by an anonymous reviewer who performed numerical simulations including a small gyro bias term and discovered satisfactory performance by the proposed attitude estimator.

(d) If $\boldsymbol{z}_v(0)$ is on the surface of sphere but not on the equator, then $\boldsymbol{z}_v(t)$ will converge to one of the poles. More precisely, if $\boldsymbol{z}_v(0)$ is in the Northern hemisphere, then $\boldsymbol{z}_v(t)$ converges to the North pole. Likewise for $\boldsymbol{z}_v(0)$ in the Southern hemisphere, the vector part of the quaternion estimation error $\boldsymbol{z}_v(t)$ converges to the South pole.

# 5. Numerical Simulations

In this section, we present results from numerical simulation of the attitude estimation algorithm. We restrict attention to the three-dimensional case so as to mimic the problem of estimating the attitude of a satellite stationed in a geosynchronous orbit. Accordingly, the reference input signal $\boldsymbol{r}(t)$, and angular velocity of the satellite $\boldsymbol{\omega}$ expressed in body frame are taken as follows

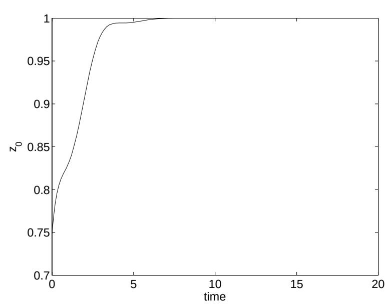
$$\boldsymbol{r}(t) = [\sin t, \cos t, 0]^T, \quad \boldsymbol{\omega} = [0, 0, 1]^T. \tag{45}$$

For all our simulations, the initial value of the quaternion representing the true/unknown attitude matrix $C(0)$ according to Eq. (28) is fixed as $\boldsymbol{q}(0) = [0.5, 0.5, 0.5, 0.5]^T$.

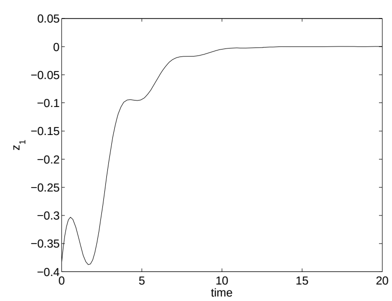
Three different simulations are performed with $\gamma = 1$. By changing $\gamma$ value, we can adjust the convergence speed of the attitude estimator. For the first case, the the initial condition for the quaternion estimate $\hat{\boldsymbol{q}}(0)$ parameterizing the attitude estimate matrix $\hat{C}(0)$ is selected to be

$$\hat{\boldsymbol{q}}(0) = [\sqrt{0.9}, \sqrt{0.1/3}, \sqrt{0.1/3}, \sqrt{0.1/3}]^T \tag{46}$$

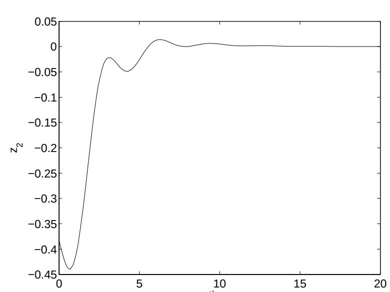
For this choice, it is easy to verify that $\hat{\boldsymbol{q}}(0) \notin \Psi_u$ and therefore, $z_o(0) \neq 0$. Given that the reference input $\boldsymbol{r}(t)$ satisfies $\dot{\boldsymbol{r}}(t) \neq 0$ and based on the discussion from previous section, we know that the attitude estimation error $\tilde{C}(t)$ asymptotically converges to zero. Simulation results for this case are presented in Fig. 3 where it can be seen, quite expectedly, that $\lim_{t\to\infty} \boldsymbol{z}_v(t) = [z_1(t), z_2(t), z_3(t)] = 0$.



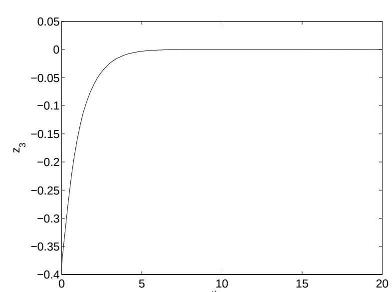

(a) The trajectory of $z_o(t)$.




(b) The trajectory of $z_1(t)$.



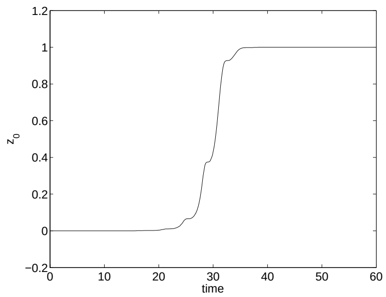

(c) The trajectory of $z_2(t)$.



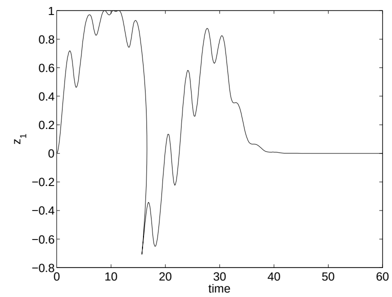

(d) The trajectory of $z_3(t)$.

Figure 3: The trajectory of the error quaternion $\boldsymbol{z}(t)$ as a function of time $t$. The initial condition $\hat{\boldsymbol{q}}(0) \notin \Psi_u$ so that $z_o(0) \neq 0$.

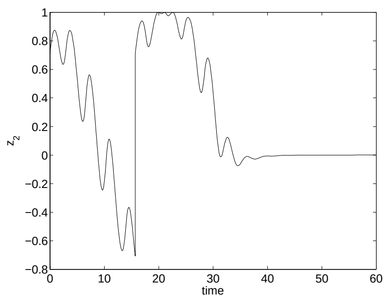
As a second case for our simulation studies, we select a different value for the initial condition for the estimate quaternion $\hat{\boldsymbol{q}}(0)$. This time the estimation starts from $\hat{\boldsymbol{q}}(0) = [0.5\sqrt{2}, 0, -0.5\sqrt{2}, 0]^T \in \Psi_u$. Obviously, for this choice, we have $z_o(0) = 0$ and therefore theoretically speaking, we should have $\{z_o(t) = 0, \|\boldsymbol{z}_v(t)\| = 1\}$ for all $t > 0$. However, from the fact that $z_o = 0$ is an unstable equilibrium manifold, we see in Fig. 4 that the vector part of the error quaternion $\boldsymbol{z}_v(t)$ converges to nearly zero after about 40 seconds of simulation time.



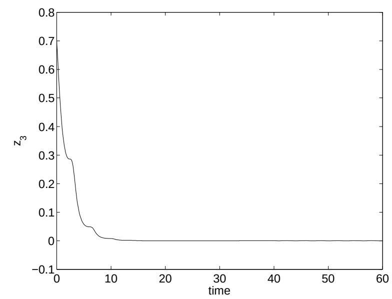

(a) The trajectory of $z_o(t)$.




(b) The trajectory of $z_1(t)$.




(c) The trajectory of $z_2(t)$.




(d) The trajectory of $z_3(t)$.

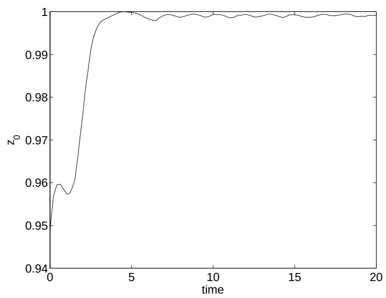
Figure 4: The trajectory of the error quaternion $\boldsymbol{z}(t)$ as a function of time $t$. The initial condition $\hat{\boldsymbol{q}}(0) \in \Psi_u$ (the unstable equilibrium manifold) so that $z_o(0) = 0$. Accumulation of numerical round-off errors ultimately results in regulation of the attitude estimation error to zero.

Finally, the measurement noise is introduced to illustrate the robustness properties of the attitude estimation algorithm. The bounded measurement noise $\boldsymbol{v}(t)$ has uniform random distribution on the interval $[-0.05, 0.05]$. The initial value of the estimate quaternion $\hat{\boldsymbol{q}}(0)$ is again taken according to Eq. (46) so that $\hat{\boldsymbol{q}}(0) \notin \Psi_u$. Simulation results including the effects of measurement noise are shown in Fig. 5. As discussed earlier, we no longer have the asymptotic convergence of error quaternion $\boldsymbol{z}_v(t)$ to zero. Instead, each component of the estimation error quaternion remains bounded so that ultimately the magnitude of $\|\boldsymbol{z}_v(t)\|$ is dictated by the magnitude of the measurement noise.

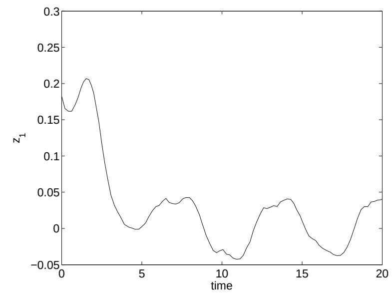
# 6. Concluding Remarks

This paper presents an adaptive estimation algorithm, together with a convergence proof that holds for all $n \times n$ proper orthogonal matrices. The importance of this problem, particularly in three-dimensions, lies in the fact that no attitude representation is ever directly measured. Instead, such information always needs to be reconstructed through techniques presented in this paper while using available input-output observations. In practice, measurements from sensors that generate input-output type vectors may be used in conjunction with the proposed methodology to reliably reconstruct the attitude information.

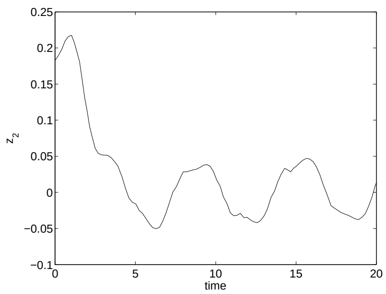
The results obtained here have broader implications, particularly for purposes of generating feedback control signals on the group of rigid body motions wherein signals corresponding to attitude representations (Euler angles, Gibbs vector, and/or quaternions) are estimated on the basis of measurement models.



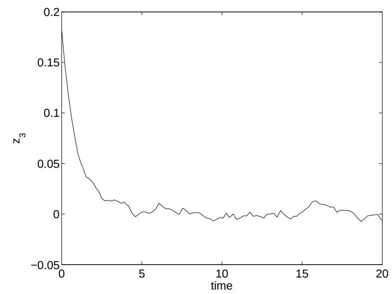

(a) The trajectory of $z_o(t)$.




(b) The trajectory of $z_1(t)$.




(c) The trajectory of $z_2(t)$.




(d) The trajectory of $z_3(t)$.

Figure 5: The trajectory of the error quaternion $\boldsymbol{z}(t)$ as a function of time $t$. The initial condition $\hat{\boldsymbol{q}}(0) \notin \Psi_u$ Measurement noise is introduced as a signal having an uniform distribution between -0.05 and 0.05.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