A NEW METHOD TO INTRODUCE A PRIORI INFORMATION IN QUEST

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The solution of the Wahba problem, a nonlinear least-squares performance index for attitude determination, does not naturally extend to include initial conditions. A multiplicative way of introducing *a priori* information into the Wahba performance index is shown.

INTRODUCTION

The determination of the spacecraft attitude from sensor measurements utilizing batch and recursive filtering methods is an important practical problem in engineering and has been extensively studied and continues to be a topic of interest [1] -[22]. The quaternion estimation algorithm (QUEST) is an implementation of Davenport's least-squares solution to the attitude determination problem [23] -[25]. In general, the least-squares batch estimation architecture employs a linearized measurement model and possesses the ability to process a variety of sensor measurement types. QUEST is a least-squares solution that avoids the linearization process leading to an optimal attitude estimate. One disadvantage of the QUEST algorithm is that it requires vector-type measurements. In a spacecraft scenario, measurements are often available in vector form, such as the magnetometer sensor output. Other sensors, such as a star tracker, can provide information that can be translated into three-dimensional unit vectors. In fact, today many star trackers output attitude quaternions as the measurements. Measurements that contain only scalar information cannot be processed by QUEST. Measurements of this kind might include the cosine of the angle between a spacecraft-fixed vector and an inertial-fixed vector. Nevertheless, the QUEST algorithm enjoys a popular following and has been employed successfully in a variety of settings.

The attitude determination problem addressed by the QUEST algorithm is to find the best overlap of a number of reference and observations vectors [23]. In this work the attitude is represented with four-element quaternions, $\overline{\mathbf{q}} = \left[\mathbf{q}^{\mathrm{T}}\ q\right]^{\mathrm{T}}$ where $\mathbf{q} = \left[q_1\ q_2\ q_3\right]^{\mathrm{T}}$ is the vector component of the quaternion and q is the scalar component. Given a set of n reference vectors, denoted by $\mathbf{r}_i \in \Re^3$, $i = 1, \ldots, n$, that are in known directions (such as directions to particular stars) in the given reference coordinate frame and a set of n observation vectors, denoted by $\mathbf{y}_i \in \Re^3$,

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$i = 1, \ldots, n$, given in a spacecraft body-fixed reference frame (such as the sensor frame), the quaternion formulation of the Wahba problem is to find the quaternion $\overline{\mathbf{q}}$ that minimizes

$$\mathcal{J}(\overline{\mathbf{q}}) = \sum_{i=1}^{n} w_i \|\mathbf{y}_i - \mathbf{T}(\overline{\mathbf{q}})\mathbf{r}_i\|_2^2 \qquad (1)$$
$$\text{subject to } \|\overline{\mathbf{q}}\|_2 = 1$$

where $\mathbf{T}(\overline{\mathbf{q}})$ is the 3×3 direction cosine matrix representing the transformation from the reference coordinate frame to the body-fixed coordinate frame. In terms of the quaternion, the transformation matrix $\mathbf{T}(\overline{\mathbf{q}})$ is given by

$$\mathbf{T}(\overline{\mathbf{q}}) = (q^2 - \mathbf{q}^{\mathrm{T}}\mathbf{q})\mathbf{I}_{3\times 3} + 2\mathbf{q}\mathbf{q}^{\mathrm{T}} - 2q[\mathbf{q}\times] \qquad (2)$$

where the vector product matrix $[\mathbf{w}\times]$ is defined such that

$$[\mathbf{w}\times]\mathbf{v} = \mathbf{w} \times \mathbf{v}$$

for all vectors $\mathbf{v} \in \Re^3$ and $\mathbf{w} \in \Re^3$. In Eq. (1), the positive scalars w_i for $i = 1, \ldots, n$, represent the accuracy of the n observations, respectively. The algorithm to solve this constrained minimization problem is due to Davenport [26]. Examining the performance index, it is observed that *a priori* information of the initial orientation of the body-fixed frame does not factor in. Incorporation of *a priori* attitude information into the Wahba performance index without significant modification of Davenport's algorithm is the subject of this work.

The remainder of this paper is organized as follows: first, the original solution to the quaternion Wahba problem is reviewed and two existing methods to incorporate *a priori* attitude information are discussed. The new proposed algorithm to incorporate *a priori* attitude information is then presented. It is shown that appropriately modifying the performance index in a multiplicative manner leads to a simple method to incorporate *a priori* attitude information within the existing structure of the Davenport's algorithm. The paper concludes with a numerical example.

DAVENPORT'S ORIGINAL SOLUTION

Davenport's original solution to the Wahba problem formulated for the quaternion is given by Keat [26]. Minimization of Wahba performance index in Eq. (1) is equivalent to maximization of the performance index

$$\mathcal{J}^{\star}(\overline{\mathbf{q}}) = \sum_{i=1}^{n} w_i^2 \mathbf{y}_i^{\mathrm{T}} \mathbf{T}(\overline{\mathbf{q}})\mathbf{r}_i \qquad (3)$$
$$\text{subject to } \|\overline{\mathbf{q}}\|_2 = 1$$

Defining the 3×3 matrix $\mathbf{B}$ as

$$\mathbf{B} \triangleq \sum_{i=1}^{n} w_i^2 \mathbf{y}_i \mathbf{r}_i^{\mathrm{T}} \qquad (4)$$

and using matrix trace properties, it follows that Eq. (3) can be written as

$$\mathcal{J}^\star(\overline{\mathbf{q}}) = \text{trace}\left[\mathbf{T}(\overline{\mathbf{q}})\mathbf{B}^{\mathrm{T}}\right] \tag{5}$$

The matrix $\mathbf{B}$ can be found recursively using

$$\mathbf{B}_k = \mathbf{B}_{k-1} + w_k \mathbf{y}_k \mathbf{r}_k^{\mathrm{T}} \tag{6}$$

where $1 \leq k \leq n$,

$$\mathbf{B}_0 \triangleq \mathbf{O} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_n$$

Denote the vector component of the quaternion by $\mathbf{q}$ and the scalar element of the quaternion by q. Substituting Eq. (2) in the performance index in Eq. (5) and using $\mathbf{B}$ in Eq. (4) yields

$$\mathcal{J}^\star = \sigma(q^2 - \mathbf{q}^{\mathrm{T}}\mathbf{q}) + 2\mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{q} - 2q\,\text{trace}\left[[\mathbf{q}\times]\mathbf{B}^{\mathrm{T}}\right]$$

where

$$\sigma \triangleq \text{trace}(\mathbf{B})$$

This problem constitutes a quadratic program, i.e. the performance index can be rewritten as

$$\mathcal{J}^\star(\overline{\mathbf{q}}) = \overline{\mathbf{q}}^{\mathrm{T}}\mathbf{K}\overline{\mathbf{q}} \tag{7}$$

where the 4×4 matrix $\mathbf{K}$ is now obtained. Define the symmetric matrix $\mathbf{S}$ as

$$\mathbf{S} \triangleq \mathbf{B} + \mathbf{B}^{\mathrm{T}}$$

Then it follows that

$$2\mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{q} = \mathbf{q}^{\mathrm{T}}\mathbf{S}\,\mathbf{q}$$

Notice that

$$-2\,\text{trace}\left[[\mathbf{q}\times]\mathbf{B}^{\mathrm{T}}\right] = -2\,\text{trace}\left[\sum_{i=1}^{n} w_i[\mathbf{q}\times]\mathbf{r}_i\mathbf{y}_i^{\mathrm{T}}\right] = 2\sum_{i=1}^{n} w_i\left(\mathbf{y}_i \times \mathbf{r}_i\right)^{\mathrm{T}}\mathbf{q}$$

Therefore, matrix $\mathbf{K}$ in Eq. (7) is given by

$$\mathbf{K} = \begin{bmatrix} \mathbf{S} - \sigma\mathbf{I}_{3\times3} & \mathbf{z} \\ \mathbf{z}^{\mathrm{T}} & \sigma \end{bmatrix} \tag{8}$$

where

$$\mathbf{z} \triangleq \sum_{i=1}^{n} w_i\left(\mathbf{y}_i \times \mathbf{r}_i\right)$$

Adjoining the constraint $\overline{\mathbf{q}}^{\mathrm{T}}\overline{\mathbf{q}} = 1$ to Eq. (7) with a Lagrange multiplier, denoted by λ, the first-order optimal condition is given by the eigenvalue problem

$$\mathbf{K}\overline{\mathbf{q}} = \lambda\overline{\mathbf{q}} \tag{9}$$

Also using Eq. (7) and Eq. (9), the performance index can be shown to be

$$\mathcal{J}^\star = \lambda$$

where λ is any of the eigenvalues of $\mathbf{K}$. Since the performance index is to be maximized, the optimal Lagrange multiplier is given by the maximum eigenvalue of $\mathbf{K}$ given in Eq. (8), and the optimal quaternion is given by the corresponding unit eigenvector. There is no need to calculate the eigenvector. The vector of Rodrigues parameters is given by

$$\mathbf{g} = \mathbf{q}/q$$

The first three rows of Eq. (9) can be expanded to be

$$(\mathbf{S} - \sigma \mathbf{I}_{3\times 3})\,\mathbf{q} + \mathbf{z}q = \lambda \mathbf{q}$$

from which the optimal Gibbs vector is found to be

$$\mathbf{g} = [(\sigma + \lambda)\mathbf{I}_{3\times 3} - \mathbf{S}]^{-1}\mathbf{z} \tag{10}$$

The optimal quaternion is given by

$$\overline{\mathbf{q}} = \frac{1}{\sqrt{1 + \mathbf{g}^{\mathrm{T}}\mathbf{g}}} \begin{bmatrix} \mathbf{g} \\ 1 \end{bmatrix}$$

Shuster and Oh [23] show how to handle Eq. (10) when matrix $(\sigma + \lambda)\mathbf{I}_{3\times 3} - \mathbf{S}$ is singular, and a computationally efficient method to determine the eigenvalue, known as QUEST.

EXISTING METHODS OF INCORPORATING A PRIORI INFORMATION

As originally posed, the Wahba problem does not include initial conditions. At least two different approaches have been proposed to add initial conditions to QUEST[27, 28]. Both solutions are given in the context of reformulating QUEST from a batch estimation algorithm to a filter formulation with state updates followed by state propagation. The recursive algorithm can be obtained from the above references or using REQUEST [29]. A brief summary of filter QUEST and extended QUEST are now presented.

Filter QUEST [27] introduces initial conditions using key properties of the Fisher information matrix. Under certain assumptions on the distribution of the measurements, the Fisher information matrix $\mathbf{F}$ for the three-dimensional representation of the estimation error is [2]

$$\mathbf{F} = \mathrm{trace}\left[\mathbf{T}(\overline{\mathbf{q}}^{TRUE})\left(\mathbf{B}^{TRUE}\right)^{\mathrm{T}}\right]\mathbf{I}_{3\times 3} - \mathbf{T}(\overline{\mathbf{q}}^{TRUE})\left(\mathbf{B}^{TRUE}\right)^{\mathrm{T}} \tag{11}$$

where $\mathbf{B}^{TRUE}$ is calculated using $\mathbf{T}(\overline{\mathbf{q}}^{TRUE})\mathbf{r}_i$ instead of the measured vector $\mathbf{y}_i$. The Fisher information matrix is asymptotically equal to the inverse of the estimation error covariance. Solving for $\mathbf{B}$ in Eq. (11), the *a priori* information $\overline{\mathbf{q}}^{\,-}$ with the associated *a priori* three-dimensional covariance $\mathbf{P}_0$ can be incorporated into the attitude estimate employing

$$\mathbf{B}_0 = \left[\frac{1}{2}\,\mathrm{trace}(\mathbf{P}_0^{-1})\mathbf{I}_{3\times 3} - \mathbf{P}_0^{-1}\right]\mathbf{T}(\overline{\mathbf{q}}^{\,-}) \tag{12}$$

as the initial condition in the recursion given in Eq. (6). It can be shown that

$$[\mathbf{z}\times] = \mathbf{B}^{\mathrm{T}} - \mathbf{B}$$

Therefore, $\mathbf{z}$ in Eq. (8) can be readily initialized since it can be calculated directly using $\mathbf{B}$. In short, the filter QUEST algorithm consists of initializing $\mathbf{B}$ with Eq. (12), then updating with Eq. (6) and propagating with

$$\mathbf{B}_{k+1}^{-} = \alpha \boldsymbol{\Phi}_k \mathbf{B}_k^{+}$$

where $\alpha \in [0, 1]$ is a heuristic parameter that approximates the degradation due to process noise as fading memory. The state transition matrix $\boldsymbol{\Phi}_k$ is found by integrating from t_k to t_{k+1} the following matrix differential equation

$$\frac{d}{dt}\boldsymbol{\Phi}_k = [\boldsymbol{\omega}\times]\boldsymbol{\Phi}_k, \quad \boldsymbol{\Phi}_k(t_k) = \mathbf{I}_{3\times 3}$$

where $\boldsymbol{\omega}$ is the body angular velocity. The attitude estimate can be extracted at any time from $\mathbf{B}$ using the QUEST algorithm.

Filter QUEST makes two assumptions:

1. The relationship between the Fisher information matrix and the covariance holds true only asymptotically, while it is assumed to be true at the initial time.
2. The relationship in Eq. (11) should contain the true rotation matrix $\mathbf{T}(\overline{\mathbf{q}}^{TRUE})$, while its initial estimate is used.

Extended QUEST [28] combines an extended Kalman filter with a recursive QUEST algorithm to estimate the attitude quaternion as well as other states, such as position, velocity, and biases. The state propagation is accomplished using standard propagation methods associated with Kalman filtering. However, in extended QUEST, the state update is performed in two steps. First, the filter states are determined as a function of the attitude quaternion, then the attitude quaternion is computed by employing a generalized quadratic programming strategy. A full derivation of extended QUEST is beyond the scope of this section, however it is relevant to this work to see how extended QUEST handles initial conditions. For ease of discussion, we assume that there is no auxiliary state vector, that is, only the attitude quaternion is being estimated. Under this assumption, the extended QUEST cost function reduces to

$$\mathcal{J}_1(\overline{\mathbf{q}}^{+}) = \left(\overline{\mathbf{q}}^{+} - \overline{\mathbf{q}}^{-}\right)^{\mathrm{T}} \mathbf{R} \left(\overline{\mathbf{q}}^{+} - \overline{\mathbf{q}}^{-}\right) + \sum_{i=1}^{n} w_i \|\mathbf{y}_i - \mathbf{T}(\overline{\mathbf{q}}^{+})\mathbf{r}_i\|_2^2$$

where $\mathbf{R}$ is a symmetric positive definite weight of the *a priori* estimate. Extended QUEST introduces the *a priori* information in the standard batch estimation way [30], by adding the weighted Euclidean distance between the *a priori* and *a posteriori* estimates to the least-squares performance index. The cost function $\mathcal{J}_1$ constitutes a generalized quadratic performance index, that is, it can be rewritten in the form

$$\mathcal{J}_{GEN}(\overline{\mathbf{q}}) = \overline{\mathbf{q}}^{\mathrm{T}}\mathbf{K}\overline{\mathbf{q}} + \mathbf{k}^{\mathrm{T}}\overline{\mathbf{q}}$$
$$\text{subject to } \|\overline{\mathbf{q}}\|_2 = 1$$

where

$$\mathbf{k} = -2\mathbf{R}^{\mathrm{T}}\overline{\mathbf{q}}^{\,-}, \qquad \mathbf{K} = \begin{bmatrix} \mathbf{S} - \sigma\mathbf{I}_{3\times3} & \mathbf{z} \\ \mathbf{z}^{\mathrm{T}} & \sigma \end{bmatrix} + \mathbf{R}$$

with $\mathbf{S}$, $\mathbf{z}$, and σ defined as in the original QUEST algorithm.

Readers interested in filter QUEST and extended QUEST should refer to the original literature for more detailed explanations. Both methods provide a clear path to incorporating *a priori* attitude information using reformulations of QUEST. Our objective is to be able to incorporate initial attitude estimates into QUEST without employing limiting assumptions or changing the fundamental character of the batch QUEST algorithm.

PROBLEM FORMULATION

To facilitate introducing initial conditions in QUEST, it is necessary to include in the performance index the distance between the *a priori* information (given by the initial conditions) and the estimated quaternion to be determined. In this work, the distance between the *a priori* and *a posteriori* attitude estimates is expressed using a quaternion-of-rotation instead of using the Euclidean distance as in extended QUEST. Benefits and drawbacks of the multiplicative and additive representations have been extensively studied, for example Pittelkau [31] shows that the additive approach can contain a bias in the estimate.

Define $\mathbf{p}$ as the quaternion representing the rotation from the *a priori* to the *a posteriori* attitude estimation so that

$$\overline{\mathbf{p}} = \begin{bmatrix} \mathbf{p}^{\mathrm{T}} & p \end{bmatrix}^{\mathrm{T}} = \overline{\mathbf{q}}^{\,+} \otimes (\overline{\mathbf{q}}^{\,-})^{-1} \tag{13}$$

where the quaternion product $\otimes$ is defined such that the quaternions are multiplied in the same order as the attitude matrices.

The norm of the vector component of the quaternion is the sine of half the angle between the *a priori* and *a posteriori* attitude representations, and therefore is a good candidate as a measure of distance. The new performance index to minimize is given by

$$\mathcal{J}_2(\overline{\mathbf{q}}^{\,+}) = w_0\|\mathbf{p}\|_2^2 + \sum_{i=1}^{n} w_i\|\mathbf{y}_i - \mathbf{T}(\overline{\mathbf{q}}^{\,+})\mathbf{r}_i\|_2^2 \tag{14}$$

The performance index $\mathcal{J}_2$ in Eq. (14) can be expressed in a more convenient form by noticing that

$$\|\mathbf{p}\|_2^2 = \frac{1}{8}\sum_{i=1}^{3} \|\mathbf{e}_i - \mathbf{T}(\overline{\mathbf{p}})\mathbf{e}_i\|_2^2$$

where $\mathbf{e}_i$ are the three vectors forming the canonical base. The proof of the identity follows. Let $\mathbf{v} \in \Re^3$ be an arbitrary vector. Expressing the rotation matrix in quaternion form, it follows that

$$\mathbf{v} - \mathbf{T}(\overline{\mathbf{p}})\mathbf{v} = 2\big(p\mathbf{I}_{3\times3} - [\mathbf{p}\times]\big)[\mathbf{p}\times]\mathbf{v} \tag{15}$$

The square of the norm of Eq. (15) is

$$\|\mathbf{v} - \mathbf{T}(\overline{\mathbf{p}})\mathbf{v}\|_2^2 = 4\mathbf{v}^{\mathrm{T}}[\mathbf{p}\times]^{\mathrm{T}}\left\{(1 - \mathbf{p}^{\mathrm{T}}\mathbf{p})\mathbf{I}_{3\times 3} - [\mathbf{p}\times]^2\right\}[\mathbf{p}\times]\mathbf{v} \tag{16}$$

Using the identity

$$\mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I}_{3\times 3} + [\mathbf{p}\times]^2 = \mathbf{p}\mathbf{p}^{\mathrm{T}}$$

Eq. (16) reduces to

$$\|\mathbf{v} - \mathbf{T}(\overline{\mathbf{p}})\mathbf{v}\|_2^2 = 4\mathbf{v}^{\mathrm{T}}[\mathbf{p}\times]^{\mathrm{T}}[\mathbf{p}\times]\mathbf{v} = 4\|\mathbf{p}\times\mathbf{v}\|_2^2$$

Choosing $\mathbf{v} = \mathbf{e}_1 = [1\ 0\ 0\ ]^{\mathrm{T}}$, it follows that

$$\|\mathbf{e}_1 - \mathbf{T}(\overline{\mathbf{p}})\mathbf{e}_1\|_2^2 = 4p_2^2 + 4p_3^2$$

Following a similar process with $\mathbf{v} = \mathbf{e}_2 = [0\ 1\ 0\ ]^{\mathrm{T}}$ and $\mathbf{v} = \mathbf{e}_3 = [0\ 0\ 1\ ]^{\mathrm{T}}$ yields

$$\sum_{i=1}^{3}\|\mathbf{e}_i - \mathbf{T}(\overline{\mathbf{p}})\mathbf{e}_i\|_2^2 = 8\|\mathbf{p}\|_2^2$$

From Eq. (13) it follows that

$$\mathbf{T}(\overline{\mathbf{p}}) = \mathbf{T}(\overline{\mathbf{q}}^{+})\left(\mathbf{T}(\overline{\mathbf{q}}^{-})\right)^{\mathrm{T}}$$

Suppose we append the "observations" $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ to the n given observations. Then, for $i = 1, 2$, and 3, we have

$$\begin{aligned}\mathbf{y}_{n+i} &= \mathbf{e}_i\\ \mathbf{r}_{n+i} &= \left(\mathbf{T}(\overline{\mathbf{q}}^{-})\right)^{\mathrm{T}}\mathbf{e}_i\\ w_{n+i} &= w_0/8\end{aligned}$$

Therefore, $\mathcal{J}_2$ can be conveniently rewritten as

$$\mathcal{J}_2(\overline{\mathbf{q}}^{+}) = \sum_{i=1}^{n+3} w_i\|\mathbf{y}_i - \mathbf{T}(\overline{\mathbf{q}}^{+})\mathbf{r}_i\|_2^2$$

which can be solved using the original QUEST solution. If this scheme is used in the absence of measurements, it will return as the *a posteriori* estimate the *a priori* value.

Notice that each component of $\mathbf{p}$ could be weighted differently by selecting appropriate values for w_{n+1}, w_{n+2}, and w_{n+3}. Also a symmetric positive definite weight $\mathbf{W}_0$ could enter the performance index as $\mathbf{p}^{\mathrm{T}}\mathbf{W}_0\mathbf{p}$ by simply appending three more observations $\mathbf{e}_4$, $\mathbf{e}_5$, and $\mathbf{e}_6$ with appropriate weights.

$$\mathbf{e}_4 = [0\ \sqrt{2}\ \sqrt{2}]^{\mathrm{T}}, \quad \mathbf{e}_5 = [\sqrt{2}\ 0\ \sqrt{2}\ ]^{\mathrm{T}}, \quad \mathbf{e}_6 = [\sqrt{2}\ \sqrt{2}\ 0\ ]$$

NUMERICAL EXAMPLE

Consider the situation depicted in Figure 1. The reference coordinate frame is represented by the vectors $\mathbf{x}_I, \mathbf{y}_I$, and $\mathbf{z}_I$. Using a roll, pitch, and yaw attitude rotation sequence, the body reference frame is oriented at $\varphi = 10$ deg, $\theta = -45$ deg, and $\psi = 60$ deg, where the angles are body-fixed frame to reference frame. The body-fixed coordinate frame is represented by the vectors $\mathbf{x}_B, \mathbf{y}_B$, and $\mathbf{z}_B$. The transformation from the reference coordinate frame to the body-fixed frame is represented by the quaternion

$$\overline{\mathbf{q}}^{TRUE} = \begin{bmatrix} -0.2603 & 0.2899 & -0.4891 & 0.7804 \end{bmatrix}^{\mathrm{T}}$$

There are five target points available for measurement, as depicted in Figure 1. The vectors to the five targets (in the reference coordinate frame) are:

$$\mathbf{r}_1 = r_1 \begin{bmatrix} 0.9962 \\ 0 \\ 0.0872 \end{bmatrix} \qquad \mathbf{r}_2 = r_2 \begin{bmatrix} 0.4924 \\ 0.8529 \\ 0.1736 \end{bmatrix} \qquad \mathbf{r}_3 = r_3 \begin{bmatrix} -0.9962 \\ 0 \\ 0.0872 \end{bmatrix}$$

$$\mathbf{r}_4 = r_4 \begin{bmatrix} 0.4532 \\ -0.7849 \\ 0.4226 \end{bmatrix} \qquad \mathbf{r}_5 = r_5 \begin{bmatrix} -0.4330 \\ -0.7500 \\ 0.5000 \end{bmatrix}$$

The targets are located at ranges $r_1 = 100$m, $r_2 = 10$m, $r_3 = 150$m, $r_4 = 75$m, and $r_5 = 50$m. The observations are not unit vectors like in the QUEST measurement model. Relative positions to the targets are measured. To generate the observations, the reference vectors were transformed into the body-fixed reference frame (using the true quaternion) and additive (normally-distributed $\mathrm{N}(0,\sigma_i^2)$, $\sigma_i = \|\mathbf{r}_i\|_2/50$) noise was applied to each element of the resulting vector. Since the weights represent the

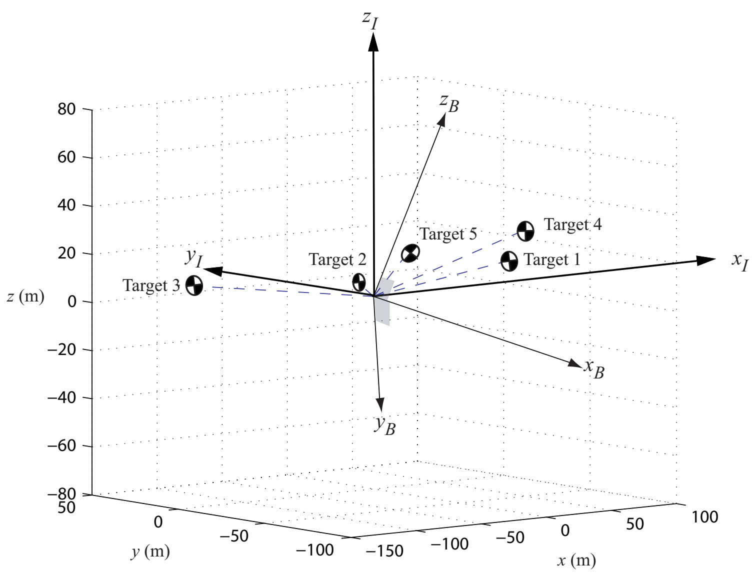

Figure 1: Reference frame and body-fixed frame showing five targets.

accuracy of the measurements, they are chosen in the standard way

$$w_i = \frac{1}{\sigma_i^2} \qquad i = 1:5$$

The weight w_0 represent the accuracy of the *a priori* information. To test the incorporation of initial conditions into QUEST, we generate estimated initial attitude errors that are zero-mean, normally-distributed with $\sigma_0 = 5$ degrees. The performance index contains sines of half angles, which are approximated by half angles. Following the standard procedure to select w_0 it would be expected that a good choice for the weight would be

$$w_0 = \frac{1}{(\sigma_0/2)^2} = \frac{1}{(0.0873/2\ rad)^2} = 525.28$$
$$w_{5+i} = w_0/8 \qquad i = 1:3$$

A Monte Carlo analysis was conducted using 500 runs, each with a different initial attitude error and measurement error, to verify the performance of the QUEST algorithm with *a priori* information. For each set of measurements and initial conditions, the algorithm was run using different values of w_0. The results are shown in Figure 2. It can be seen that even a poor initial estimate will reduce the error, and that the best performance is indeed obtain for $w_0 \simeq 525$. In fact the numerical optimum was found for $1/\sqrt{w_0} = 2.588\ deg$, very close to the predicted $2.5\ deg$. Notice however that the result is due to the distribution used. By using normal distributions and choosing the weights as the inverses of the variances, the estimate is the maximum likelihood estimate. For other distributions, choosing the weights as the inverses of the variances is a good rule of thumbs but generally leads to no optimal solution, neither in a maximum likelihood sense nor in a minimum mean square error sense, which

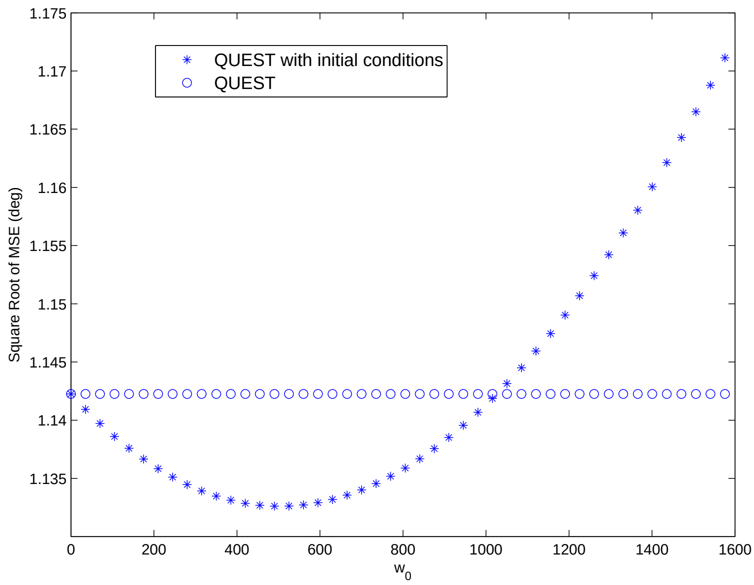

Figure 2: Mean square errors for QUEST and QUEST with initial condition as a function of σ_0.

in general produce different estimators. Therefore, the weights are design parameters that need to be tuned according to the specific problem.

If the *a priori* estimate is over-weighted, i.e. w_0 increases, the estimate degrades. As w_0 decreases, the *a priori* information is de-weighted to the extent that the results approach the QUEST algorithm without initial information. Also, it can be seen that in general the QUEST algorithm with *a priori* information performs better under the conditions in this numerical experiment.

CONCLUSION

A simple way of introducing *a priori* information in QUEST was presented. The advantage of this new method is that no approximations are needed and that the *a priori* information is included in a multiplicative fashion which leads to a simple physical interpretation, and allows for easy determination on the weight for the *a priori* estimate. Like in the linear least-squares approach used for batch estimation, weights are introduced and they represent the accuracy of the measurements and *a priori* estimate. In linear least-squares, choosing the weights as the inverses of the covariances results in the minimum variance estimate. The non-linearities of the Wahba problem however, prevent such result. Assuming very specific error distributions, the solutions to both the the linear least-squares and the Wahba problem can be interpreted as maximum likelihood estimates.

Assuming the measurements are distributed as in Shuster[2], making a small angle approximation for the *a priori* distribution, and making other assumptions on the Fisher information matrix, filter QUEST shows how to introduce initial conditions, under those circumstances the proposed approach reduces to filter QUEST when the initial weight is chosen in the same way. However, this work does not make any limiting assumptions in its derivation, and is therefore valid for all measurement distributions, and for arbitrarily large initial estimation errors.

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