Kalman Filters with Uncompensated Biases

Renato Zanetti*

The Charles Stark Draper Laboratory, Houston, Texas, 77058

Robert H. Bishop†

Marquette University, Milwaukee, WI 53201

I. INTRODUCTION

An underlying assumption of the Kalman filter is that the measurement and process disturbances can be accurately modeled as random white noise. Various mitigation strategies are available when this assumption is invalid. In practice sensor errors are often modeled more accurately as the sum of a white noise component and a strongly correlated component. The correlated components can, for example, be random constant biases. In this case, a standard technique is to augment the Kalman filter state vector and estimate the random biases. In an attempt to decouple the bias estimation from the state estimation, Friedland [1] estimated the state as though the bias was not present, and then added the contribution of the bias. Friedland showed that this approach is equivalent to augmenting the state vector. This technique, known as two-stage Kalman filtering or separate-bias Kalman estimation, was then extended to incorporate a walk in the bias forced by white noise [2]. To account for the bias walk, the process noise covariance was increased heuristically, and optimality conditions were derived [3, 4].

In this work, the effects of the noise and biases are considered as sources of uncertainty and not as elements of the state vector. This approach is applicable in situations when the biases are not observable or when there is not enough information to discern the biases from the measurements. A common approach would be to tune the filter using process and measurement noise such that the sample covariance obtained through Monte Carlo analysis matches the filter state estimation error covariance. The technique presented here takes

*Senior Member of the Technical Staff, Vehicle Dynamics and Control, 17629 El Camino Real, Suite 470, rzanetti@draper.com, Member AIAA.

†Dean of Engineering, Marquette University, P.O. Box 1881, Milwaukee, WI 53201-1881, robert.bishop@marquette.edu, Fellow AIAA.

advantage of the structure of the biases to obtain a more precise representation of their contributions to the state estimation uncertainty. The resulting algorithms are useful in quantifying the uncertainty in a single simulation along the nominal state trajectory. This process can aid in tuning the filter as well as be employed onboard to obtain an accurate measure of the uncertainty of the state estimates.

The approach taken is similar to that of the consider filter proposed by Schmidt [5]. The consider filter can be applied to our problem and the two solution approaches, although different in form, are functionally the same.

The goal of this paper is to introduce the uncompensated bias Kalman filter as presented by the authors in [6] and [7]. More recently Hough [8] independently derived a similar algorithm and applied it to orbit determination. This technical note shows the relation between these two recent techniques as well as their equivalency to the Schmidt consider filter [5].

II. DISCRETE KALMAN FILTER WITH UNCOMPENSATED BIAS

Consider the stochastic dynamical system model

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{\Upsilon}_k \mathbf{b}_\nu + \mathbf{J}_k \boldsymbol{\nu}_k, \tag{1}$$

where $\mathbf{x}_k \in \Re^n$ is the state vector at time t_k, $\mathbf{\Phi}_k \in \Re^{n\times n}$ is the deterministic state transition matrix $\mathbf{\Phi}_k = \mathbf{\Phi}(t_{k+1}, t_k)$, and $\boldsymbol{\nu}_k \in \Re^r$ is the process noise assumed to be a zero-mean, white noise vector sequence with

$$\mathrm{E}\left\{\boldsymbol{\nu}_k\right\} = \mathbf{0} \ \ \forall\, k, \qquad \mathrm{E}\left\{\boldsymbol{\nu}_k\, \boldsymbol{\nu}_j^{\mathrm{T}}\right\} = \mathbf{V}_k\, \delta_{kj},$$

where $\mathbf{V}_k \in \Re^{r\times r}$, $\mathbf{V}_k \geq \mathbf{0}$ for all k, and $\delta_{kj} = 1$ if $k = j$, and $\delta_{kj} = 0$ if $k \neq j$. A random constant vector bias $\mathbf{b}_\nu \in \Re^m$ is also considered with the assumed properties that

$$\mathrm{E}\left\{\mathbf{b}_\nu\right\} = \mathbf{0}, \qquad \mathrm{E}\left\{\mathbf{b}_\nu\, \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{B}_\nu, \qquad \mathrm{E}\left\{\boldsymbol{\nu}_k \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{0} \ \ \forall k$$

where $\mathbf{B}_\nu \in \Re^{m\times m}$ and $\mathbf{B}_\nu > \mathbf{0}$. The shape matrices $\mathbf{\Upsilon}_k \in \Re^{n\times m}$ and $\mathbf{J}_k \in \Re^{n\times r}$ are deterministic. From Eq. (1) and the fact that $\boldsymbol{\nu}_k$ is modeled as a zero-mean random vector sequence and $\mathbf{b}_\nu$ is modeled as a zero-mean random constant vector, an unbiased estimate of the state $\hat{\mathbf{x}}_{k-1}$ can be propagated forward in time to obtain an unbiased estimate at time t_k via

$$\hat{\mathbf{x}}_k^- = \mathbf{\Phi}_{k-1} \hat{\mathbf{x}}_{k-1}^+,$$

where $\hat{\mathbf{x}}_k^-$ is the state estimate at t_k before a measurement update and $\hat{\mathbf{x}}_{k-1}^+$ is the state estimate after the measurement update at t_{k-1}. The estimation error at t_k before the measurement update is defined as

$$\mathbf{e}_k^- = \mathbf{x}_k - \hat{\mathbf{x}}_k^-, \tag{2}$$

and the estimation error at t_k after the measurement update is defined as

$$\mathbf{e}_k^+ = \mathbf{x}_k - \hat{\mathbf{x}}_k^+. \tag{3}$$

At t_k, it is assumed that measurements are available in the form

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \boldsymbol{\Lambda}_k \mathbf{b}_\eta + \boldsymbol{\eta}_k, \tag{4}$$

where $\mathbf{y}_k \in \Re^p$, $\mathbf{H}_k$ is the measurement mapping matrix, $\boldsymbol{\eta}_k \in \Re^p$ is the measurement noise assumed to be a zero-mean, white noise vector sequence with

$$\mathrm{E}\left\{\boldsymbol{\eta}_k\right\} = \mathbf{0} \;\; \forall\, k, \qquad \mathrm{E}\left\{\boldsymbol{\eta}_k\, \boldsymbol{\eta}_j^{\mathrm{T}}\right\} = \mathbf{R}_k\, \delta_{kj},$$

where $\mathbf{R}_k \in \Re^{p\times p}$, $\mathbf{R}_k > \mathbf{0}$ for all k. A random constant vector bias $\mathbf{b}_\eta \in \Re^q$ is also considered with the assumed properties that

$$\mathrm{E}\left\{\mathbf{b}_\eta\right\} = \mathbf{0}, \qquad \mathrm{E}\left\{\mathbf{b}_\eta\, \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{B}_\eta, \qquad \mathrm{E}\left\{\boldsymbol{\eta}_k \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{0} \;\; \forall k$$

where $\mathbf{B}_\eta \in \Re^{q\times q}$ and $\mathbf{B}_\eta > \mathbf{0}$. The shape matrix $\boldsymbol{\Lambda}_k \in \Re^{n\times q}$ is deterministic. We also assume that

$$\mathrm{E}\left\{\mathbf{b}_\nu \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{0}, \mathrm{E}\left\{\boldsymbol{\eta}_j \boldsymbol{\nu}_k^T\right\} = \mathbf{0}, \mathrm{E}\left\{\boldsymbol{\nu}_k \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{0}, \text{ and } \mathrm{E}\left\{\boldsymbol{\eta}_k \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{0} \text{ for all } k, j.$$

The propagated estimation error after the measurement update at t_{k-1} to before the next measurement update at t_k is

$$\mathbf{e}_k^- = \boldsymbol{\Phi}_{k-1} \mathbf{e}_{k-1}^+ + \boldsymbol{\Upsilon}_{k-1} \mathbf{b}_\nu + \mathbf{J}_{k-1} \boldsymbol{\nu}_{k-1}, \tag{5}$$

and the associated covariance propagation, $\mathbf{P}_k^- = \mathrm{E}\left\{\mathbf{e}_k^- \mathbf{e}_k^{-\mathrm{T}}\right\}$, is given by

$$\begin{aligned}\mathbf{P}_k^- &= \boldsymbol{\Phi}_{k-1} \mathbf{P}_{k-1}^+ \boldsymbol{\Phi}_{k-1}^{\mathrm{T}} + \boldsymbol{\Upsilon}_{k-1} \mathbf{B}_\nu \boldsymbol{\Upsilon}_{k-1}^{\mathrm{T}} + \mathbf{J}_{k-1} \mathbf{V}_{k-1} \mathbf{J}_{k-1}^{\mathrm{T}} \\ &\quad + \boldsymbol{\Phi}_{k-1}\, \mathrm{E}\left\{\mathbf{e}_{k-1}^+ \mathbf{b}_\nu^{\mathrm{T}}\right\} \boldsymbol{\Upsilon}_{k-1}^{\mathrm{T}} + \boldsymbol{\Upsilon}_{k-1}\, \mathrm{E}\left\{\mathbf{b}_\nu (\mathbf{e}_{k-1}^+)^{\mathrm{T}}\right\} \boldsymbol{\Phi}_{k-1}^{\mathrm{T}},\end{aligned} \tag{6}$$

where we assume that $\boldsymbol{\nu}_{k-1}$ is uncorrelated to $\mathbf{b}_\nu$ and $\mathbf{e}^+_{k-1}$. The state update at t_k is assumed to be the linear update

$$\hat{\mathbf{x}}^+_k = \hat{\mathbf{x}}^-_k + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k), \tag{7}$$

where

$$\hat{\mathbf{y}}_k = \mathbf{H}_k\hat{\mathbf{x}}_k,$$

and $\hat{\mathbf{y}}_k$ follows from the measurement model in Eq. (4) and the fact that $\boldsymbol{\eta}_k$ is modeled as a zero-mean random vector sequence and $\mathbf{b}_\eta$ is modeled as a zero-mean random constant vector. The update in Eq. (7) provides an unbiased *a posteriori* estimate when the *a priori* estimate is unbiased. With the estimation error at t_k after the measurement update defined as in Eq. (3), we obtain the estimation error after the update as

$$\mathbf{e}^+_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{e}^-_k - \mathbf{K}_k\boldsymbol{\Lambda}_k\mathbf{b}_\eta - \mathbf{K}_k\boldsymbol{\eta}_k. \tag{8}$$

The update of the state estimation error covariance, $\mathbf{P}^+_k = \mathrm{E}\left\{\mathbf{e}^+_k\mathbf{e}^{+^{\mathrm{T}}}_k\right\}$, is given by

$$\begin{aligned}\mathbf{P}^+_k = {} & (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}^-_k(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k\boldsymbol{\Lambda}_k\mathbf{B}_\eta\boldsymbol{\Lambda}^{\mathrm{T}}_k\mathbf{K}^{\mathrm{T}}_k + \mathbf{K}_k\mathbf{R}_k\mathbf{K}^{\mathrm{T}}_k + \\ & - (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\,\mathrm{E}\left\{\mathbf{e}^-_k\mathbf{b}^{\mathrm{T}}_\eta\right\}\boldsymbol{\Lambda}^{\mathrm{T}}_k\mathbf{K}^{\mathrm{T}}_k - \mathbf{K}_k\boldsymbol{\Lambda}_k\,\mathrm{E}\left\{\mathbf{b}_\eta(\mathbf{e}^-_k)^{\mathrm{T}}\right\}(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)^{\mathrm{T}},\end{aligned} \tag{9}$$

where we assume that $\boldsymbol{\eta}_k$ is uncorrelated to $\mathbf{b}_\eta$ and $\mathbf{e}^-_k$. Substituting Eq. (5) into Eq. (8) yields

$$\mathbf{e}^+_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\left(\boldsymbol{\Phi}_{k-1}\mathbf{e}^+_{k-1} + \boldsymbol{\Upsilon}_{k-1}\mathbf{b}_\nu + \mathbf{J}_{k-1}\boldsymbol{\nu}_{k-1}\right) - \mathbf{K}_k\boldsymbol{\Lambda}_k\mathbf{b}_\eta - \mathbf{K}_k\boldsymbol{\eta}_k.$$

Forming $\mathbf{e}^+_k\,\mathbf{b}^{\mathrm{T}}_\nu$ and taking the expectation yields

$$\mathrm{E}\left\{\mathbf{e}^+_k\,\mathbf{b}^{\mathrm{T}}_\nu\right\} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\left[\boldsymbol{\Phi}_{k-1}\,\mathrm{E}\left\{\mathbf{e}^+_{k-1}\,\mathbf{b}^{\mathrm{T}}_\nu\right\} + \boldsymbol{\Upsilon}_{k-1}\mathbf{B}_\nu\right]. \tag{10}$$

where we assume that $\mathbf{b}_\nu$ is uncorrelated to $\mathbf{b}_\eta$, $\boldsymbol{\nu}_{k-1}$, and $\boldsymbol{\eta}_k$, $\forall k$. Factor $\mathrm{E}\left\{\mathbf{e}^+_{k-1}\,\mathbf{b}^{\mathrm{T}}_\nu\right\}$ as

$$\mathrm{E}\left\{\mathbf{e}^+_{k-1}\,\mathbf{b}^{\mathrm{T}}_\nu\right\} = \mathbf{L}_{k-1}\mathbf{B}_\nu, \tag{11}$$

where we note that $\mathbf{B}_\nu > \mathbf{0}$. Substituting Eq. (11) into Eq. (6) yields

$$\begin{aligned}\mathbf{P}^-_k = {} & \boldsymbol{\Phi}_{k-1}\mathbf{P}^+_{k-1}\boldsymbol{\Phi}^{\mathrm{T}}_{k-1} + \boldsymbol{\Upsilon}_{k-1}\mathbf{B}_\nu\boldsymbol{\Upsilon}^{\mathrm{T}}_{k-1} + \mathbf{J}_{k-1}\mathbf{V}_{k-1}\mathbf{J}^{\mathrm{T}}_{k-1} \\ & + \boldsymbol{\Phi}_{k-1}\mathbf{L}_{k-1}\mathbf{B}_\nu\boldsymbol{\Upsilon}^{\mathrm{T}}_{k-1} + \boldsymbol{\Upsilon}_{k-1}\mathbf{L}_{k-1}\mathbf{B}^{\mathrm{T}}_\nu\boldsymbol{\Phi}^{\mathrm{T}}_{k-1},\end{aligned} \tag{12}$$

where $\mathbf{L}_k$ is found via recursion. Substituting Eq. (11) into Eq. (10) yields

$$\mathrm{E}\left\{\mathbf{e}_k^+ \mathbf{b}_\nu^\mathrm{T}\right\} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\left[\mathbf{\Phi}_{k-1}\mathbf{L}_{k-1} + \mathbf{\Upsilon}_{k-1}\right]\mathbf{B}_\nu = \mathbf{L}_k\mathbf{B}_\nu.$$

Recalling that $\mathbf{B}_\nu > \mathbf{0}$, we obtain the recursion for $\mathbf{L}_k$ as

$$\mathbf{L}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)(\mathbf{\Phi}_{k-1}\mathbf{L}_{k-1} + \mathbf{\Upsilon}_{k-1}). \tag{13}$$

The initial condition, $\mathbf{L}_0$, for the $\mathbf{L}_k$ recursion is found by considering the state estimation error after the first update, or

$$\mathbf{e}_1^+ = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\left(\mathbf{\Phi}_0\mathbf{e}_0 + \mathbf{\Upsilon}_0\mathbf{b}_\nu + \mathbf{J}_0\boldsymbol{\nu}_0\right) - \mathbf{K}_1\mathbf{\Lambda}_1\mathbf{b}_\eta - \mathbf{K}_1\boldsymbol{\eta}_1.$$

Computing $\mathbf{e}_1^+ \mathbf{b}_\nu^\mathrm{T}$ and taking the expectation yields

$$\mathrm{E}\left\{\mathbf{e}_1^+ \mathbf{b}_\nu^\mathrm{T}\right\} = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\,\Upsilon_0\mathbf{B}_\nu,$$

with the assumption that $\mathbf{b}_\nu$ is not correlated with the initial state estimation error, $\mathbf{e}_0$. We find that

$$\mathbf{L}_1 = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\mathbf{\Upsilon}_0,$$

which can be obtained using the recursion of Eq. (13) for $k = 1$ if we set $\mathbf{L}_0 = \mathbf{0}$. Therefore, we start the recursion for $\mathbf{L}_k$ with $\mathbf{L}_0 = \mathbf{0}$.

The estimation error at t_{k+1} before the measurement update is found from Eq. (5) to be

$$\mathbf{e}_{k+1}^- = \mathbf{\Phi}_k\mathbf{e}_k^+ + \mathbf{\Upsilon}_k\mathbf{b}_\nu + \mathbf{J}_k\boldsymbol{\nu}_k. \tag{14}$$

Substituting Eq. (8) into Eq. (14) yields the recurrence relation

$$\mathbf{e}_{k+1}^- = \mathbf{\Phi}_k\left[(\mathbf{I} - \mathbf{K}_k\,\mathbf{H}_k)\,\mathbf{e}_k^- - \mathbf{K}_k\,\boldsymbol{\eta}_k - \mathbf{K}_k\,\mathbf{\Lambda}_k\,\mathbf{b}_\eta\right] + \mathbf{\Upsilon}_k\mathbf{b}_\nu + \mathbf{J}_k\boldsymbol{\nu}_k.$$

Forming $\mathbf{e}_{k+1}^- \mathbf{b}_\eta^\mathrm{T}$ and taking the expectation, it follows that

$$\mathrm{E}\left\{\mathbf{e}_{k+1}^- \mathbf{b}_\eta^\mathrm{T}\right\} = \mathbf{\Phi}_k\,(\mathbf{I} - \mathbf{K}_k\,\mathbf{H}_k)\,\mathrm{E}\left\{\mathbf{e}_k^- \mathbf{b}_\eta^\mathrm{T}\right\} - \mathbf{\Phi}_k\mathbf{K}_k\,\mathbf{\Lambda}_k\mathbf{B}_\eta, \tag{15}$$

where we assume that $\mathbf{b}_\eta$ is uncorrelated to $\mathbf{b}_\nu$, $\boldsymbol{\nu}_k$, and $\boldsymbol{\eta}_k$, $\forall k$. Factoring $\mathrm{E}\left\{\mathbf{e}_k^- \mathbf{b}_\eta^\mathrm{T}\right\}$ as

$$\mathrm{E}\left\{\mathbf{e}_k^- \mathbf{b}_\eta^\mathrm{T}\right\} = \mathbf{M}_k\,\mathbf{B}_\eta, \tag{16}$$

it follows that

$$\mathrm{E}\left\{\mathbf{e}_{k+1}^{-}\,\mathbf{b}_{\eta}^{\mathrm{T}}\right\} = \mathbf{M}_{k+1}\,\mathbf{B}_{\eta}, \tag{17}$$

and using Eq. (15)–(17), the matrix $\mathbf{M}_k$ can be found recursively as

$$\mathbf{M}_{k+1} = \mathbf{\Phi}_k\left[(\mathbf{I}-\mathbf{K}_k\mathbf{H}_k)\mathbf{M}_k - \mathbf{K}_k\mathbf{\Lambda}_k\right]. \tag{18}$$

If, at the initial time, a propagation occurs such that

$$\mathbf{e}_1^{-} = \mathbf{\Phi}_0\mathbf{e}_0 + \mathbf{\Upsilon}_0\mathbf{b}_{\nu} + \mathbf{J}_0\boldsymbol{\nu}_0,$$

with the assumption that $\mathbf{b}_\eta$ is not correlated with the initial state estimation error, $\mathbf{e}_0$, it follows from Eq. (16) that $\mathrm{E}\left\{\mathbf{e}_1^{-}\,\mathbf{b}_{\eta}^{\mathrm{T}}\right\} = \mathbf{0}$ which in turn implies that $\mathbf{M}_1 = \mathbf{0}$ since $\mathbf{B}_\eta > \mathbf{0}$. We assume here that a propagation occurs at t_0 before the first measurement update at t_1. Hence, we require the starting values $\mathbf{L}_0 = \mathbf{0}$ and $\mathbf{M}_1 = \mathbf{0}$. If an update occurs at time t_0 before the first propagation, the same algorithm can be used by setting $\mathbf{M}_0 = \mathbf{0}$ and $\mathbf{L}_0 = \mathbf{0}$. Substituting Eq. (16) into Eq. (9), after some rearrangement, we obtain

$$\begin{aligned}\mathbf{P}_k^{+} = \mathbf{P}_k^{-} &- \mathbf{K}_k\big(\mathbf{H}_k\mathbf{P}_k^{-} + \mathbf{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\big) - \big(\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k\,\mathbf{B}_\eta\mathbf{\Lambda}_k^{\mathrm{T}}\big)\mathbf{K}_k^{\mathrm{T}} + \\ &+ \mathbf{K}_k\big(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{\Lambda}_k\mathbf{B}_\eta\mathbf{\Lambda}_k^{\mathrm{T}} + \mathbf{H}_k\mathbf{M}_k\,\mathbf{B}_\eta\mathbf{\Lambda}_k^{\mathrm{T}} + \mathbf{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}\big)\mathbf{K}_k^{\mathrm{T}}.\end{aligned} \tag{19}$$

Taking the derivative of the trace of $\mathbf{P}_k^{+}$ with respect to $\mathbf{K}_k$, setting the result to zero, and solving for $\mathbf{K}_k$ yields the optimal gain,

$$\mathbf{K}_k = \big(\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k\,\mathbf{B}_\eta\mathbf{\Lambda}_k^{\mathrm{T}}\big)\mathbf{W}_k^{-1}, \tag{20}$$

where the matrix $\mathbf{W}_k$ is the covariance of the residuals given by

$$\mathbf{W}_k = \mathrm{E}\left\{(\mathbf{y}-\hat{\mathbf{y}})\,(\mathbf{y}-\hat{\mathbf{y}})^{\mathrm{T}}\right\} = \mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{\Lambda}_k\mathbf{B}_\eta\mathbf{\Lambda}_k^{\mathrm{T}} + \mathbf{H}_k\mathbf{M}_k\,\mathbf{B}_\eta\mathbf{\Lambda}_k^{\mathrm{T}} + \mathbf{\Lambda}_k\mathbf{B}_\eta\mathbf{M}_k^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}.$$

Substituting Eq. (20) into Eq. (19) yields

$$\mathbf{P}_k^{+} = \mathbf{P}_k^{-} - \mathbf{K}_k\mathbf{W}_k\mathbf{K}_k^{\mathrm{T}}.$$

The discrete uncompensated bias algorithm is summarized in Table 1. The uncompensated bias algorithm for the continuous time models of the measurements and dynamics is presented in [7].

System Model	$\mathbf{x}_{k+1} = \boldsymbol{\Phi}_k \mathbf{x}_k + \boldsymbol{\Upsilon}_k \mathbf{b}_\nu + \mathbf{J}_k \boldsymbol{\nu}_k$
	$\mathrm{E}\left\{\boldsymbol{\nu}_k\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\boldsymbol{\nu}_k \boldsymbol{\nu}_j^{\mathrm{T}}\right\} = \mathbf{V}_k \, \delta_{kj}, \forall k, j$
	$\mathrm{E}\left\{\mathbf{b}_\nu\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\mathbf{b}_\nu \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{B}_\nu > \mathbf{O}$
Measurement Model	$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \boldsymbol{\Lambda}_k \mathbf{b}_\eta + \boldsymbol{\eta}_k$
	$\mathrm{E}\left\{\boldsymbol{\eta}_k\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\boldsymbol{\eta}_k \boldsymbol{\eta}_j^{\mathrm{T}}\right\} = \mathbf{R}_k \delta_{kj}, \forall k, j$
	$\mathrm{E}\left\{\mathbf{b}_\eta\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\mathbf{b}_\eta \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{B}_\eta > \mathbf{O}$
Assumptions	$\mathrm{E}\left\{\boldsymbol{\nu}_k \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\boldsymbol{\eta}_k \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\boldsymbol{\eta}_k \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{O}, \ \forall k$
	$\mathrm{E}\left\{\mathbf{b}_\nu \, \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\boldsymbol{\nu}_k \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{O}, \quad \mathrm{E}\left\{\boldsymbol{\eta}_k \boldsymbol{\nu}_j^{\mathrm{T}}\right\} = \mathbf{O}, \ \forall k, j$
Initial Conditions	$\hat{\mathbf{x}}_0^+ = \mathrm{E}\left\{\mathbf{x}(t_0)\right\}, \quad \mathbf{P}_0^+ = \mathrm{E}\left\{\mathbf{e}_0 \mathbf{e}_0^{\mathrm{T}}\right\}, \quad \mathbf{e}_0 = \mathbf{x}(t_0) - \hat{\mathbf{x}}_0^+$
Recursion Initialization	$\mathbf{M}_1 = \mathbf{O}, \quad \mathbf{L}_0 = \mathbf{O}$
State Propagation	$\hat{\mathbf{x}}_k^- = \boldsymbol{\Phi}_{k-1} \hat{\mathbf{x}}_{k-1}^+, \quad k = 1, 2, \cdots$
Covariance Propagation	$\mathbf{P}_k^- = \boldsymbol{\Phi}_{k-1} \mathbf{P}_{k-1}^+ \boldsymbol{\Phi}_{k-1}^{\mathrm{T}} + \boldsymbol{\Upsilon}_{k-1} \mathbf{B}_\nu \boldsymbol{\Upsilon}_{k-1}^{\mathrm{T}} + \mathbf{J}_{k-1} \mathbf{V}_{k-1} \mathbf{J}_{k-1}^{T} +$
	$+ \boldsymbol{\Phi}_{k-1} \mathbf{L}_{k-1} \mathbf{B}_\nu \boldsymbol{\Upsilon}_{k-1}^{\mathrm{T}} + \boldsymbol{\Upsilon}_{k-1} \mathbf{B}_\nu \mathbf{L}_{k-1}^{\mathrm{T}} \boldsymbol{\Phi}_{k-1}^{\mathrm{T}}$
Gain Calculation	$\mathbf{K}_k = \left(\mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k \, \mathbf{B}_\eta \boldsymbol{\Lambda}_k^{\mathrm{T}}\right) \mathbf{W}_k^{-1}$
	$\mathbf{W}_k = \mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \boldsymbol{\Lambda}_k \mathbf{B}_\eta \boldsymbol{\Lambda}_k^{\mathrm{T}} + \mathbf{H}_k \mathbf{M}_k \, \mathbf{B}_\eta \boldsymbol{\Lambda}_k^{\mathrm{T}} + \boldsymbol{\Lambda}_k \mathbf{B}_\eta \mathbf{M}_k^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}}$
State Update	$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k (\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^-)$
Covariance Update	$\mathbf{P}_k^+ = \mathbf{P}_k^- - \mathbf{K}_k \mathbf{W}_k \mathbf{K}_k^{\mathrm{T}}$
L Calculation	$\mathbf{L}_k = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)(\boldsymbol{\Phi}_{k-1} \mathbf{L}_{k-1} + \boldsymbol{\Upsilon}_{k-1})$
M Calculation	$\mathbf{M}_{k+1} = \boldsymbol{\Phi}_k \left[(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{M}_k - \mathbf{K}_k \boldsymbol{\Lambda}_k\right]$

Table 1. Discrete-time Kalman filter with uncompensated bias.

III. Equivalency with Bias Characterization Filter

In this section we show that the bias characterization filter algorithm presented by Hough [8] is a subset of the discrete uncompensated bias algorithm previously presented by the authors [6, 7]. The bias characterization filter incorporates only biases in the discrete measurements. In our notation, this is equivalent to stating that $\mathbf{B}_\nu = \mathbf{O}$. In the derivation by Hough, the estimation error $\delta\hat{\mathbf{x}}$ is defined with the opposite sign of the estimation errors in Eq. (2) and (3), or

$$\delta\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k = -\mathbf{e}_k. \tag{21}$$

Other notation equivalencies include the correlation matrix $\mathbf{S}_k \leftrightarrow -\mathbf{M}_k \mathbf{B}_\eta$, measurement mapping matrix $\mathbf{C}_k \leftrightarrow \mathbf{H}_k$, the bias shaping matrix $\mathbf{M}_k \leftrightarrow \boldsymbol{\Lambda}_k$, and the prior correlation matrix is already propagated to the current measurement time, such that the state transition matrix in Eq. (18) reduces to the identity matrix, or $\mathbf{I} \leftrightarrow \mathbf{\Phi}_k$. With the above changes our discrete uncompensated bias filter can be rewritten as

$$
\begin{aligned}
\mathbf{K}_k &= \mathbf{L}_k^{\mathrm{T}} \mathbf{N}_k^{-1} \\
\mathbf{L}_k &= \mathbf{C}_k \mathbf{P}_k^- - \mathbf{M}_k (\mathbf{S}_k^-)^{\mathrm{T}} \\
\mathbf{N}_k &= \mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k \mathbf{B}_\eta \mathbf{M}_k^{\mathrm{T}} + \mathbf{R}_k - \mathbf{H}_k \mathbf{S}_k^- \mathbf{M}_k^{\mathrm{T}} - \mathbf{M}_k (\mathbf{S}_k^-)^{\mathrm{T}} \mathbf{H}_k^{\mathrm{T}} \\
\mathbf{P}_k^+ &= \mathbf{P}_k^- - \mathbf{K}_k \mathbf{L}_k - \mathbf{L}_k^{\mathrm{T}} \mathbf{K}_k^{\mathrm{T}} + \mathbf{K}_k \mathbf{N}_k \mathbf{K}_k^{\mathrm{T}} \\
\mathbf{S}^+ &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{S}_k^- + \mathbf{K}_k \mathbf{M}_k \mathbf{B}_k,
\end{aligned}
\tag{22}
$$

which are the equations presented by Hough [8].

The next section shows the mathematical equivalency between our uncompensated bias filter and the consider filter. Since the bias compensation filter [8] is a subset of the uncompensated bias filter, this implies that the bias compensation filter is also equivalent to the consider filter.

IV. Equivalency with Consider Filter

Slightly different versions of the consider filter exist. In this work we refer to the consider filter developed by Schmidt and presented by Jazwinski [5].

Under the same modeling assumptions of Section II, assume that an augmented state $\mathbf{z}_k$ is created at t_k by

$$
\mathbf{z}_k = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{b}_\nu \\ \mathbf{b}_\eta \end{bmatrix}. \tag{23}
$$

Denoting the augmented state estimation error at t_k after the measurement update as $\mathbf{e}_{z,k}^+ = \mathbf{z}_k - \hat{\mathbf{z}}_k^+$, we have the augmented state error covariance matrix at t_k after the measurement update given by

$$
\mathbf{Z}_k^+ = \mathrm{E}\left\{\mathbf{e}_{z,k}^+ \mathbf{e}_{z,k}^{+\mathrm{T}}\right\}
$$

where the nonzero submatrices are

$$
\begin{aligned}
&\mathrm{E}\left\{\mathbf{e}_k^+ \mathbf{e}_k^{+\mathrm{T}}\right\} = \mathbf{P}_k^+, \quad \mathrm{E}\left\{\mathbf{e}_k^+ \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{L}_k \mathbf{B}_\nu \\
&\mathrm{E}\left\{\mathbf{e}_k^+ \mathbf{b}_\eta^{\mathrm{T}}\right\} = \left[(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{M}_k - \mathbf{K}_k \mathbf{\Lambda}_k\right] \mathbf{B}_\eta \\
&\mathrm{E}\left\{\mathbf{b}_\nu \mathbf{b}_\nu^{\mathrm{T}}\right\} = \mathbf{B}_\nu, \quad \mathrm{E}\left\{\mathbf{b}_\eta \mathbf{b}_\eta^{\mathrm{T}}\right\} = \mathbf{B}_\eta
\end{aligned}
\tag{24}
$$

Denote the augmented state estimation error at t_{k+1} before the measurement update as

$\mathbf{e}^-_{z,k+1} = \mathbf{z}_{k+1} - \hat{\mathbf{z}}^-_{k+1}$ and the associated augmented state error covariance matrix as

$$\mathbf{Z}^-_{k+1} = \mathrm{E}\left\{\mathbf{e}^-_{z,k+1}\mathbf{e}^{-^{\mathrm{T}}}_{z,k+1}\right\}.$$

The augmented state estimation error covariance is propagated from t_k to t_{k+1} via

$$\mathbf{Z}^-_{k+1} = \mathbf{\Psi}_k\mathbf{Z}^+_k\mathbf{\Psi}^{\mathrm{T}}_k + \mathbf{W}_k \tag{25}$$

where the state transition matrix and process noise are given by

$$\mathbf{\Psi}_k = \begin{bmatrix} \mathbf{\Phi}_k & \mathbf{\Upsilon}_k & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{bmatrix}, \qquad \mathbf{W}_k = \begin{bmatrix} \mathbf{J}_k\mathbf{V}_k\mathbf{J}^{\mathrm{T}}_k & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix}.$$

Computing $\mathbf{Z}^-_{k+1}$ in Eq. (25) yields the following relationships

$$\begin{aligned} &\mathrm{E}\left\{\mathbf{e}^-_{k+1}\mathbf{e}^{-^{\mathrm{T}}}_{k+1}\right\} = \mathbf{\Phi}_k\mathbf{P}^+_k\mathbf{\Phi}^{\mathrm{T}}_k + \mathbf{\Phi}_k\mathbf{L}_k\mathbf{B}_\nu\mathbf{\Upsilon}^{\mathrm{T}}_k + \mathbf{\Upsilon}_k\mathbf{B}_\nu\mathbf{L}^{\mathrm{T}}_k\mathbf{\Phi}^{\mathrm{T}}_k + \mathbf{\Upsilon}_k\mathbf{B}_\nu\mathbf{\Upsilon}^{\mathrm{T}}_k + \mathbf{J}_k\mathbf{V}_k\mathbf{J}^{\mathrm{T}}_k \\ &\mathrm{E}\left\{\mathbf{e}^-_{k+1}\mathbf{b}_\nu\right\} = \left[\mathbf{\Phi}_k\mathbf{L}_k + \mathbf{\Upsilon}_k\right]\mathbf{B}_\nu, \quad \mathrm{E}\left\{\mathbf{e}^-_{k+1}\mathbf{b}_\eta\right\} = \mathbf{\Phi}_k\left[(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{M}_k - \mathbf{K}_k\mathbf{\Lambda}_k\right]\mathbf{B}_\eta \end{aligned} \tag{26}$$

We know that $\mathbf{P}^-_{k+1} = \mathrm{E}\left\{\mathbf{e}^-_{k+1}\mathbf{e}^{-^{\mathrm{T}}}_{k+1}\right\}$ and from Eq. (17) that $\mathrm{E}\left\{\mathbf{e}^-_{k+1}\,\mathbf{b}^{\mathrm{T}}_\eta\right\} = \mathbf{M}_{k+1}\,\mathbf{B}_\eta$, hence from Eq. (26) it follows that

$$\begin{aligned} \mathbf{P}^-_{k+1} &= \mathbf{\Phi}_k\mathbf{P}^+_k\mathbf{\Phi}^{\mathrm{T}}_k + \mathbf{\Phi}_k\mathbf{L}_k\mathbf{B}_\nu\mathbf{\Upsilon}^{\mathrm{T}}_k + \mathbf{\Upsilon}_k\mathbf{B}_\nu\mathbf{L}^{\mathrm{T}}_k\mathbf{\Phi}^{\mathrm{T}}_k + \mathbf{\Upsilon}_k\mathbf{B}_\nu\mathbf{\Upsilon}^{\mathrm{T}}_k + \mathbf{J}_k\mathbf{V}_k\mathbf{J}^{\mathrm{T}}_k \\ \mathbf{M}_{k+1} &= \mathbf{\Phi}_k\left[(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{M}_k - \mathbf{K}_k\mathbf{\Lambda}_k\right] \end{aligned}$$

which are the same as Eq. (12) and Eq. (18). When a measurement becomes available at t_{k+1} the augmented measurement mapping matrix is

$$\mathbf{Y}_{k+1} = \begin{bmatrix} \mathbf{H}_{k+1} & \mathbf{O} & \mathbf{\Lambda}_{k+1} \end{bmatrix},$$

the residual covariance is given by

$$\begin{aligned} \mathbf{W}_{k+1} &= \mathbf{Y}_{k+1}\mathbf{Z}^-_{k+1}\mathbf{Y}^{\mathrm{T}}_{k+1} + \mathbf{R}_{k+1} \\ &= \mathbf{H}_{k+1}\mathbf{P}^-_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{\Lambda}_{k+1}\mathbf{B}_\eta\mathbf{M}^{\mathrm{T}}_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{H}_{k+1}\mathbf{M}_{k+1}\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1} + \mathbf{\Lambda}_{k+1}\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1} + \mathbf{R}_{k+1}, \end{aligned}$$

The consider gain $\mathbf{K}^*_{k+1}$ is one in which the rows corresponding to the consider states are zero [9]

$$\mathbf{K}^*_{k+1} = \begin{bmatrix} \mathbf{K}_{k+1} \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix}. \tag{27}$$

The Schmidt-Kalman gain is obtained by minimizing the trace of $\mathbf{P}^+_{k+1}$ (and equivalently of $\mathbf{Z}^+_{k+1}$) out of all possible choices of consider gains in Eq. (27). The Schmidt-Kalman gain is given by

$$\mathbf{K}^*_{k+1} = \begin{bmatrix} \mathbf{P}^-_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{M}_{k+1}\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1} \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix} \mathbf{W}^{-1}_{k+1}, \tag{28}$$

where the non-zero rows end up being equal to the corresponding portion of the globally optimal Kalman gain

$$\mathbf{K}^{opt}_{k+1} = \begin{bmatrix} \mathbf{P}^-_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{M}_{k+1}\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1} \\ \mathbf{B}_\nu\left(\mathbf{L}^{\mathrm{T}}_k\mathbf{\Phi}^{\mathrm{T}}_k + \mathbf{\Upsilon}_k\right)\mathbf{H}^{\mathrm{T}}_{k+1} \\ \mathbf{B}_\eta\mathbf{M}^{\mathrm{T}}_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1} \end{bmatrix} \mathbf{W}^{-1}_{k+1}. \tag{29}$$

Then, computing the *a posteriori* augmented state error covariance matrix,

$$\mathbf{Z}^+_{k+1} = \mathrm{E}\left\{\mathbf{e}^+_{z,k+1}\mathbf{e}^{+\mathrm{T}}_{z,k+1}\right\},$$

via

$$\mathbf{Z}^+_{k+1} = (\mathbf{I} - \mathbf{K}^*_{k+1}\mathbf{Y}_{k+1})\mathbf{Z}^-_{k+1}(\mathbf{I} - \mathbf{K}^*_{k+1}\mathbf{Y}_{k+1})^{\mathrm{T}} + \mathbf{K}^*_{k+1}\mathbf{R}_{k+1}\mathbf{K}^{*\mathrm{T}}_{k+1},$$

with $\mathbf{K}_{k+1} = (\mathbf{P}^-_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{M}_{k+1}\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1})\mathbf{W}^{-1}_{k+1}$, $\mathbf{P}^+_{k+1} = \mathrm{E}\left\{\mathbf{e}^+_{k+1}\mathbf{e}^{+\mathrm{T}}_{k+1}\right\}$, and $\mathrm{E}\left\{\mathbf{e}^+_{k+1}\,\mathbf{b}^{\mathrm{T}}_\nu\right\} = \mathbf{L}_{k+1}\,\mathbf{B}_\nu$, yields

$$\begin{aligned}\mathbf{P}^+_{k+1} = \mathbf{P}^-_{k+1} &- \mathbf{K}_{k+1}\big(\mathbf{H}_{k+1}\mathbf{P}^-_{k+1} + \mathbf{\Lambda}_{k+1}\mathbf{B}_\eta\mathbf{M}^{\mathrm{T}}_{k+1}\big) - \big(\mathbf{P}^-_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} \\ &+ \mathbf{M}_{k+1}\,\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1}\big)\mathbf{K}^{\mathrm{T}}_{k+1} + \mathbf{K}_{k+1}\big(\mathbf{H}_{k+1}\mathbf{P}^-_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1} + \mathbf{R}_{k+1} + \mathbf{\Lambda}_{k+1}\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1} \\ &\qquad + \mathbf{H}_{k+1}\mathbf{M}_{k+1}\,\mathbf{B}_\eta\mathbf{\Lambda}^{\mathrm{T}}_{k+1} + \mathbf{\Lambda}_{k+1}\mathbf{B}_\eta\mathbf{M}^{\mathrm{T}}_{k+1}\mathbf{H}^{\mathrm{T}}_{k+1}\big)\mathbf{K}^{\mathrm{T}}_{k+1}. \\ \mathbf{L}_{k+1} = (\mathbf{I} &- \mathbf{K}_{k+1}\mathbf{H}_{k+1})\,(\mathbf{\Phi}_k\mathbf{L}_k + \mathbf{\Upsilon}_k)\end{aligned}$$

which are the same as Eq. (19) and Eq. (13). The fundamental relationships presented in Table 1 are replicated by the consider filter when we set the initial conditions as $\mathbf{M}_0 = \mathbf{0}$, $\mathbf{L}_0 = \mathbf{0}$, and $\mathbf{K}_0 = \mathbf{0}$.

V. Conclusions

In this work, the algorithms for precise navigation are derived to include uncompensated bias terms in both the process and measurement noise. The proposed algorithm treats the biases as error terms rather than states and produces the minimum variance estimator under this assumption. The proposed algorithm is compared to the well-known Schmidt-Kalman filter or consider filter. The consider filter treats the biases as states but neglects to update them when a measurement becomes available. This note shows that the two algorithms, while approaching the problem from different perspectives, are mathematically equivalent.

References

[1] Friedland, B., "Treatment of Bias in Recursive Filtering," *IEEE Transactions on Automatic Control*, Vol. 14, No. 4, August 1969, pp. 359–367.

[2] Ignagni, M. B., "Separate-Bias Kalman Estimator with Bias State Noise," *IEEE Transactions on Automatic Control*, Vol. 35, No. 3, March 1990, pp. 338–341.

[3] Alouani, A. T., Xia, P., Rice, T. R., and Blair, W. D., "On the Optimality of Two-Stage State Estimation In the Presence of Random Bias," *IEEE Transactions on Automatic Control*, Vol. 38, No. 8, August 1993, pp. 1279–1282.

[4] Ignagni, M. B., "Optimal and Suboptimal Separate-Bias Kalman Estimator for a Stochastic Bias," *IEEE Transactions on Automatic Control*, Vol. 45, No. 3, March 2000, pp. 547–551.

[5] Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Vol. 64 of *Mathematics in Sciences and Engineering*, Academic Press, 1970, pp. 281–285.

[6] Zanetti, R. and Bishop, R. H., "Entry Navigation Dead-Reckoning Error Analysis: Theoretical Foundations of the Discrete-Time Case," *Proceedings of the AAS/AIAA Astrodynamics Specialist Conference held August 19–23, 2007, Mackinac Island, Michigan*, Vol. 129 of *Advances in the Astronautical Sciences*, 2007, pp. 979–994, AAS 07-311.

[7] Zanetti, R., *Advanced Navigation Algorithms for Precision Landing*, Ph.D. thesis, The University of Texas at Austin, Austin,Texas, December 2007.

[8] Hough, M. E., "Orbit Determination with Improved Covariance Fidelity, Including Sensor Measurement Biases," *Journal of Guidance, Control, and Dynamics*, Vol. 34, No. 3, May–June 2011, pp. 903–911.

[9] Bierman, G. J., *Factorization Methods for Discrete Sequential Estimation*, Vol. 128 of *Mathematics in Sciences and Engineering*, Academic Press, 1978, pp. 165–171.