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Response to "A numerical study on tensile strength of low-density Kagome networks made of brittle fibers"

Re: On the evaluation of the energy release rate in brittle beam-lattices

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In a discussion of the work by Mane et al. (2024), Ryvkin (2025) suggested a different approach to evaluate the energy release rate in lattice materials and to derive their effective fracture toughness. He noted that Mane et al. (2024) did not take into account the energy lost due to the strain redistribution in the lattice and thus underestimated the energy release value. Here, as a response to the discussion, we present our view on the evaluation of the energy release rate and the effective fracture toughness of two-dimensional (2D) Kagome lattices as considered by Mane et al. (2024).

First, let us clarify the distinction between "energy release rate" and "critical energy release rate" in fracture mechanics. The two terms may look similar, but they are fundamentally different in our opinion. The energy approach in fracture mechanics is founded in thermodynamics, where the free energy of the system can be written as a function of the crack area (or crack length in 2D). The basic principle of thermodynamics dictates that, as an irreversible process, the crack can grow only if the free energy decreases with the crack area. The free energy of the system includes elastic energy of the body and other inelastic energy such as surface energy, plastic energy (dissipation), etc. Let Φ be the total free energy, including an elastic part (Φ_{elastic}) and an inelastic part ($\Phi_{\text{inelastic}}$), namely

$$\Phi(A_c) = \Phi_{\text{elastic}}(A_c) + \Phi_{\text{inelastic}}(A_c) \tag{1}$$

where A_c stands for the area of the crack face. The thermodynamic condition for the crack to grow requires that

$$\frac{d\Phi}{dA_c} = \frac{d\Phi_{\text{elastic}}}{dA_c} + \frac{d\Phi_{\text{inelastic}}}{dA_c} < 0 \tag{2}$$

Thus, the critical condition for crack growth is:

$$\frac{d\Phi}{dA_c} = \frac{d\Phi_{\text{elastic}}}{dA_c} + \frac{d\Phi_{\text{inelastic}}}{dA_c} = 0 \tag{3}$$

or equivalently

$$-\frac{d\Phi_{\text{elastic}}}{dA_c} = \frac{d\Phi_{\text{inelastic}}}{dA_c} \tag{4}$$

The left-hand side of Eq. (4) is the reduction of the elastic energy per unit area of crack growth, which is called the **energy release rate**. The right-hand side of Eq. (4) is the increase of the inelastic energy (dissipation) per unit area of crack growth, which is often called by different names, such as fracture energy, fracture toughness, or "**critical energy release rate**". Regardless how it is called, the above energy-based critical condition for fracture separates elastic energy release from inelastic energy dissipation. The reduction (or release) of the elastic energy in the body drives the crack to grow, which however must overcome a penalty as the inelastic energy dissipation increases with crack growth. The above critical condition is often written as

$$G = G_c \tag{5}$$

where the energy release rate, $G = -\frac{d\Phi_{\text{elastic}}}{dA_c}$, and the critical energy release rate, $G_c = \frac{d\Phi_{\text{inelastic}}}{dA_c}$.

The **energy release rate** (G) is a loading parameter, depending on the specimen and crack geometry (including crack length) as well as boundary conditions.

Under the condition of small-scale yielding or small-scale inelastic processes, the **critical energy release rate** (G_c) is a material property (fracture toughness), independent of the specimen/crack geometry or boundary conditions.

To evaluate the energy release rate of a crack, we solve a boundary value problem, often assuming a small-scale inelastic process zone near the crack tip. Whether the crack can grow or not, the energy release rate can be evaluated, which does not require the knowledge of any inelastic processes except for the assumption of a small-scale inelastic process zone.

To evaluate the critical energy release rate (or fracture toughness) of a material, by definition we would need to know exactly what *inelastic* processes are involved in fracture. For example, in the case of a brittle material like glass, Griffith (1921) simply assumed that the material (glass) is elastic all the way till fracture so that the only inelastic process is the creation of new surfaces due to crack growth. In this idealized case (for brittle materials), the critical energy release rate (or fracture toughness) is simply due to the increase of surface energy, that is

$$\frac{d\Phi_{\text{inelastic}}}{dA_c} = 2\gamma \tag{6}$$

where γ is the surface energy per unit area (for both upper and lower

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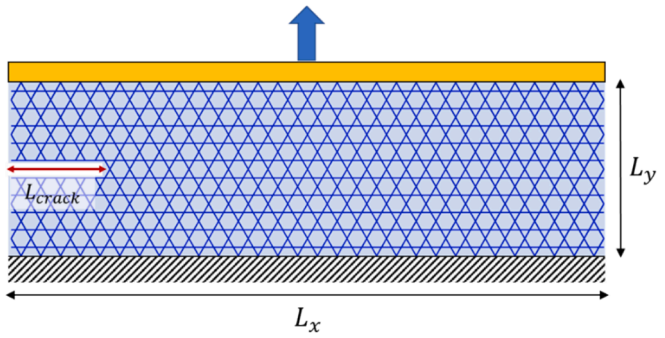

Fig. 1. A Kagome network with a crack-like defect, subject to tension with the clamped boundary condition.

crack faces). Eq. (6) gives a reasonable estimate of the critical energy release rate in the order of ~1 J/m^2 for glass. However, for metals, Eq. (6) would underestimate the critical energy release rate, because plastic deformation near the crack tip (even for small-scale yielding) is inelastic and significantly increases the energy dissipation (on top of the surface energy) per unit area of crack growth.

In practice, however, the inelastic processes associated with crack growth are often too complicated, and fracture toughness of a material is commonly measured by using a particular specimen with a crack. The energy release rate of the crack in the specimen can be calculated as a function of the applied load P, $G = G(P)$, whether the crack grows or not. At a critical load, $P = P_c$, the crack starts growing, and the corresponding energy release rate is the critical energy release rate, $G_c = G(P_c)$. This procedure relies on the above energy-based critical condition, Eq. (5), bypassing the detailed inelastic processes.

In Mane et al. (2024) Section 6.2, we considered steady-state crack growth in a Kagome network subject to tension with clamped edges (Fig. 1). For such a specimen (with a long crack), the elastic energy release rate associated with the crack growth is well known:

$$G = \frac{1}{2}\frac{E_{eff}}{1-\nu_{eff}^2}\varepsilon_y^2 L_y \tag{7}$$

where ε_y is the applied strain. Note that this energy release rate is independent of the crack length, as long as the crack is sufficiently long ($L_{crack} \gg L_y$). When the strain ε_y reaches a critical value, the crack starts growing and reaches a steady state as shown in the numerical simulations in Mane et al. (2024). The corresponding strain for the steady state was found to be $\varepsilon_{SS} \approx 0.6\varepsilon_f$, for the specimen with a relative density $\rho = 0.118$ and $L_y/L = 10\sqrt{3}$. Using this value for the applied strain ε_y in Eq. (7), we obtained the energy release rate during the steady-state crack growth:

$$G_{SS} = 0.138 E_f \varepsilon_f^2 L \tag{8}$$

Here, the effective elastic properties of the Kagome network are used: $E_{eff} \approx \frac{1}{3}\rho E_f$ and $\nu_{eff} \approx \frac{1}{3}$.

By the same critical condition for fracture as Eq. (5), the steady-state energy release rate equals the fracture toughness of the material (for steady state crack growth), namely

$$\Gamma_{SS} = G_{SS} \tag{9}$$

Here, we chose to use Γ for toughness and G for energy release rate, to distinguish the two. The above approach is commonly used to measure fracture toughness of various materials.

In the case of 2D Kagome networks, Mane et al. (2024) proposed an alternative approach to estimate the steady-state fracture toughness, Γ_{SS}, based on the near-tip analysis of the *inelastic* processes, following the definition, $\Gamma = \frac{d\Phi_{\text{inelastic}}}{dA_c}$. Here, for the Kagome networks of brittle fibers, the only inelastic process is the brittle fracture of the fiber segments

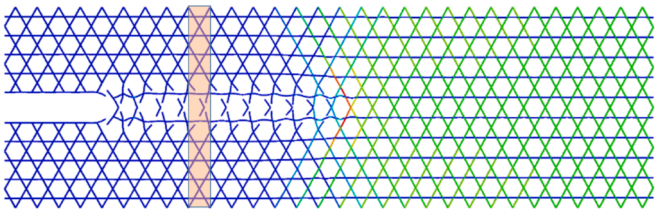

Fig. 2. Simulated steady-state damage progression (or crack growth) in a Kagome network (Mane et al., 2024).

around the crack tip. It was assumed that a crack grows in the steady state by cutting through one layer of unit cells, so that four slanted fiber segments per unit cell are fractured. This leads to Eq. (6.9) in Mane et al. (2024):

$$\Gamma_{SS} = \frac{4U_f V_f}{A_{uc}} = \frac{1}{\sqrt{3}}\rho E_f \varepsilon_f^2 L \tag{10}$$

For the specimen with $\rho = 0.118$, Eq. (10) predicts a steady-state toughness, $\Gamma_{SS} = 0.068 E_f \varepsilon_f^2 L$, which is about half of the steady-state energy release rate G_{SS} in Eq. (8). Therefore, the above approach appears to underestimate the steady-state energy release rate. This discrepancy was noted in Mane et al. (2024) and was attributed to the fact that, in the numerical simulation, more than four fiber segments per unit cell were fractured during the steady state crack growth. As shown in Fig. 2 below, besides the four fiber segments in the unit cell along the crack plane, there are four more broken fiber segments (two on each side), as the crack front passed through the shaded box. In other words, the inelastic damage process was not fully localized in one layer of unit cells along the crack plane. The damage process spread to the next layer of unit cells on both sides of the crack. Thus, instead of four, there were eight fiber segments as the crack grows by one unit cell. Consequently, the fracture toughness should be:

$$\Gamma_{SS} = \frac{8U_f V_f}{A_{uc}} = \frac{2}{\sqrt{3}}\rho E_f \varepsilon_f^2 L \tag{11}$$

which is two times of Eq. (10). For the specimen with $\rho = 0.118$, Eq. (11) predicts a steady-state toughness, $\Gamma_{SS} = 0.136 E_f \varepsilon_f^2 L$, in close agreement with the steady-state energy release rate G_{SS} in Eq. (8). Therefore, we believe that the underestimated fracture toughness by Eq. (10) is because of the underestimated dissipation due to the damage process in the broken fiber segments. To evaluate the effective fracture toughness (or the critical energy release rate) from the near-tip processes, we must take into account all the *inelastic* energy dissipation associated with crack growth.

As discussed above, our view differs from Ryvkin (2025), who suggested that evaluation of the critical energy release rate should include "energy lost in the broken elements" and "energy lost due to the strain redistribution in the lattice". In our opinion, only the "energy lost in the broken elements" should be included for calculating the critical energy release rate (or the fracture toughness), not the "energy lost due to the strain redistribution in the lattice". Strain redistribution in the lattice may reduce the elastic energy, but elastic deformation is a reversible process and does not dissipate any energy. All dissipation results from inelastic processes. In all materials, there is elastic strain redistribution around the crack as the crack grows. The reduction of the elastic energy gives the energy release rate on the left hand side of Eq. (4), while the inelastic dissipation gives the toughness on the right-hand side of Eq. (4).

A related point of discussion is on the difference between the critical energy release rate for initiation of crack growth and that for steady-state crack growth. As noted in Mane et al. (2024), damage initiation occurred in the numerical simulations at a critical strain level different from that for steady state growth. Previous works (Fleck and Qiu, 2007;

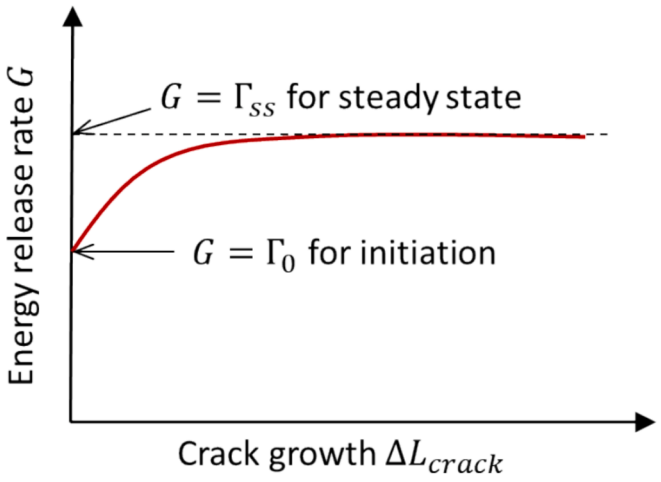

Fig. 3. A fracture resistance curve (R-curve).

Luan et al., 2022) assumed that initiation of crack growth occurs when the maximum local tensile stress at any point attains the strength of the fiber material, which means the first link at the crack tip. This assumption appears to be different from the condition for steady-state crack growth. In other words, the critical condition for initiation of crack growth can be different from that for steady-state crack growth. Such difference has been described by fracture resistance curves (R-curves, see Fig. 3) for various materials (Bao and Suo, 1992). Typically, a crack starts growing when $G = \Gamma_0$, and the energy release rate increases as the crack grows, eventually reaching a steady state value, $G = \Gamma_{ss}$. The increase of the energy release rate from Γ_0 to Γ_{ss} is due to the development of the inelastic damage process zone around the crack. In the ideal case when the damage process zone is infinitely small, we would have $\Gamma_0 = \Gamma_{ss}$. However, when the damage process zone size is finite, $\Gamma_0 < \Gamma_{ss}$.

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