Two kinds of laser sound help students comprehend dispersion

Authors

Abstract

Dispersion is a very fundamental physical concept, but few vivid and straightforward examples of dispersion can intrigue students' interests or arouse their curiosity. In this article, the author puts forward and analyzes two kinds of laser sound (whistler), laser sound in ice holes, and at slinky springs, as pedagogical examples of dispersion. On the one hand, these examples are the typical dispersive phenomenon, which is interesting and explicit enough to help students grasp dispersion physical concept; on the other hand, they are easily carried out, indicating the examples are timely and practical, since few students are permitted to enter college physics lab due to COVID-19.

I. INTRODUCTION

For wave propagation, group velocity, phase velocity, and dispersion are very common and fundamental physical concepts. In 1839, W. R. Hamilton[1] firstly proposed the idea of group velocity. Afterward, Lord Rayleigh defined and discussed group velocity, phase velocity, and dispersion very clearly.[2,3] Thanks to such excellent basic work, scientists nowadays do employ them to a diversity of scientific research fields, like underwater acoustics, seismic acoustics, biomedical ultrasonic, etc. However, although the dispersion physical concept has grown up for over one hundred years, undergraduates and even graduates still feel confused and obscure about it. Honestly speaking, I also spend a lot of time thinking over the dispersion concept. Note that we live in a dispersive medium every day, but we provide few vivid examples of dispersion from our lives to physics teaching. For example, at the class of *Theoretical Acoustics*[4], waveguide[5] and solid are characteristic dispersive medium in terms of multi-frequency wave propagation, but the pedagogy full of abstract mathematical derivation might not leave a clear physics figure in learners' mind. I do not mean that teachers would not like to show some vivid and simple examples to help students grasp the dispersion physical concept, but to find suitable-teaching examples is not easy. For example, Prof. Walter Lewin[6] in MIT uses the example of deep-water wave at the class of *Vibrations and Waves*. I don't mean this is not a good example, but the example is not audiovisual, which might leave difficulty for students to imagine the characteristics of deep-water waves.

Therefore, some vivid and audiovisual pedagogical examples of dispersion are necessary. Out of the enlightenment of sci-fiction films like STAR WARS, I realize that laser sound[7] is a typical result of dispersion suitable for physics teaching. In this article, I introduce two kinds of laser sound resulting from two dispersive ways. And after teaching practice, the pedagogical examples of laser sound help explain the dispersion physical concept.

The first example is the ice-hole laser sound from Dr. Peter Neff's Twitter post.[8] Dr. Peter Neff is a geologist who often digs ice cylinder bars from the Antarctic iceberg. Take an ice cylinder away and Leave an about one-hundred meters hole. When people drop a block of ice, whistler or laser sound is followingly produced. Up to now, more than ten million people have browsed the post. In my opinion, the intriguing phenomenon is a perfect pedagogical example of dispersion and students would like to devote themselves to answer the awesome sound out of curiosity.

Second, the well-known slinky spring toys[9] can also produce laser sound[10]. In the 1950s, the slinky toy was popular all over the United States because it could go downstairs by itself. Meanwhile, it was also a good prop in vaudeville. Afterward, in college physics, the slinky springs are generally used to display standing waves and researched on its changing of the mass center.[11,12] Prof. Frank S. Crawford noticed that the slinky would whistle when we used a pencil to tap the spring.[13-15] Nowadays, it's still necessary for us to discuss how to employ slinky springs' whistler or laser sound to exemplify dispersion.

Besides, these two pedagogical examples are timely and practical, because to get dispersive phenomenon need not professional equipment in a laboratory. As for the dispersion in ice holes, we just need to capture and analyze the audio data from the Twitter post; and the slinky springs are very easy to be purchased online or from physical toy stores. Although there are several excellent pedagogical examples of dispersion, like acoustic waveguides, capillary waves, etc.[5,16,17], they are indeed impractical at the current time, for few students are permitted to enter laboratory due to COVID-19. Therefore, dispersion in ice holes and dispersion at slinky springs are timely pedagogical examples. Note that concrete teaching ways are flexible. At our class of *Theoretical Acoustics*, the prof. Xiaodong Li arranged these examples as after-class homework and encouraged us to deliver a presentation in class, of which the professor gave the assessment. Or, if courses gear down to the low ability of students, professors may demonstrate these two examples in class, instead of requiring students to probe into those by themselves.

II. DISPERSION CONCEPT[18]

Even though the definition of dispersion is a simple and basic concept, some students in our class still have little knowledge of it. Thus, so that the pedagogical examples can work as we expect, we do need to introduce dispersion concept to students explicitly.

With the development of theoretical physics, scientists realize that few single-frequency waves in nature. Namely, the wave phenomenon is mostly in the way of multifrequency. In that way, group velocity is defined to describe the speed of combination. And phase velocity is the rate of propagation of wave state. (Figure 1 shows the simplest situation about group velocity and phase velocity.) In wave mechanics, Schrodinger called these groups or combination "wave-packets."[19] At the perspective of signal processing, group velocity is the rate of “modulation” and phase velocity is the rate of “carrier”. In certain scenarios, group velocity does not be equal to phase velocity, in which wave propagation is called dispersive or medium is named dispersive.

To clarify dispersion, a simple combination of groups obtains when two waves,

$$\begin{aligned} \omega_1 &= \omega + \Delta\omega, k_1 = k + \Delta k \\ \omega_2 &= \omega - \Delta\omega, k_2 = k - \Delta k \end{aligned} \tag{1}$$

are superimposed, giving:

$$y = A\cos(\omega_1 t - k_1 x) + A\cos(\omega_2 t - k_2 x) = 2A\cos(\Delta\omega t - \Delta k x)\cos(\omega t - kx) \tag{2}$$

in which ω means angular frequency; k means wave beam; A is the amplitude of each wavelet.

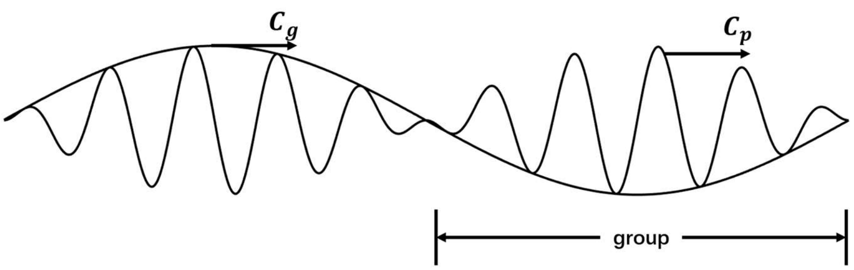

Figure 1 the sketch drawing of phase velocity and group velocity

So, we define group velocity and phase velocity as respectively

$$C_g = \frac{\Delta\omega}{\Delta k}, C_p = \frac{\omega}{k} \tag{3}$$

Furthermore, groups are more likely to be made up of infinite frequencies components. In this case,

$$C_g = \frac{\partial\omega}{\partial k}, C_p = \frac{\omega}{k} \tag{4}$$

To go further, we substitute $C_p = \frac{\omega}{k}$ into $C_g = \frac{\partial\omega}{\partial k}$, and then

$$C_g = C_p + k\frac{\partial C_p}{\partial k} = C_p - \lambda\frac{\partial C_p}{\partial \lambda} \tag{5}$$

Up to this point, let us discuss the condition of dispersion. Whether phase velocity is independent of frequency (wavelength) or not is necessary and sufficient condition of dispersion. In other words, if phase velocity is relevant to frequency, dispersion occurs; otherwise, there is no dispersive phenomenon. However, when it comes to group velocity, the situation is sort of different. If group velocity changes along with frequencies, dispersion is bound to occur; however, if group velocity is independent of frequencies, dispersion possibly or impossibly shows up. Or, if we want to use graphical language to describe dispersion, that is what Figure 2 presents.

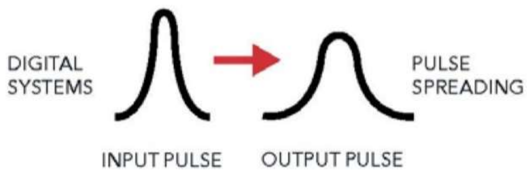

Figure 2 the graphical language of dispersion

In the time domain, if the waveform of a signal changes along with propagating, the phenomenon is “dispersion”. As for the non-physics department students, if they can build up the physical image on dispersion in their minds, it’s knowledgeable for them to work out the following two kinds of laser sound.

III. DISPERSION IN ICE HOLES

To explain the laser sound, the first step is to know what a dispersion phenomenon in ice holes is. The best way is to watch the Twitter post video from Dr. Peter Neff[8]. It is believable that students might feel amazed and weird for the first time they hear such sounds. Our ears just listen to the information of the time domain, while the information at the frequency domain is more helpful for students to analyze it physically, so the short-time Fourier transform (STFT) is essential. (Figure 3)

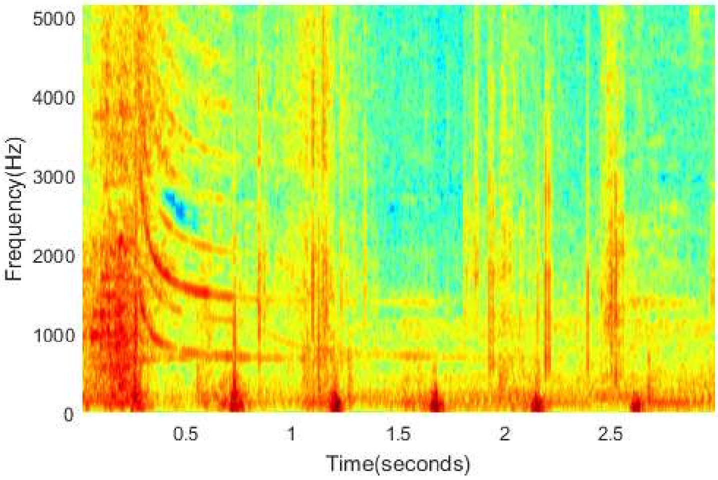

Figure 3 The time-frequency diagram of the laser sound.

According to Figure 3, it's very evident that the "Piu" laser sound is out of downward sweep frequency. To further explain, we create a cylinder hole model with a rigid wall which means we don't consider reflection loss of acoustic energy at all. When the ice chunk hit the bottom of the ice hole, it produced a pulse signal containing all kinds of frequency components. Because of the certain property of the ice hole in terms of wave propagation, the pulse signal is switched into such a "Piu" laser sound.

From Peter Neff's Twitter post[8], the diameter of the ice hole is 0.2718m and depth is 90m ($h = 90m$). And in the air at -20 degrees Celsius, the speed of sound is about 320m/s[20] ($c = 320m/s$). We build a cylinder coordinate, as Figure 9 (appendix) shows. In the cylinder hole space, wave equation[4] is

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{6}$$

And then we employ Laplace Calculator into the above wave equation and obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{7}$$

Afterward, we use the method of separation of variables and substitute boundary conditions into wave equations, leading to the dispersion equation. (The concrete steps are at the appendix.)

$$k_z=\sqrt{k^2-k_r^2}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{\omega_{mn}}{c}\right)^2} \tag{8}$$

in which $\omega = 2\pi * f$, $\omega_{mn} = 2\pi * f_{mn}$.

table 1 the cut-off frequency of different orders

$f_{mn}(Hz)$	$m=0$	$m=1$	$m=2$
$n=0$	0	690.28	1145.10
$n=1$	1436.81	1999.23	2514.04
$n=2$	2630.27	3200.57	3623.89

At this point, we can get frequency-varying and modal-varying group velocity in terms of the formula (4),

$$c_g=\frac{\partial \omega}{\partial k}=\frac{d\omega}{dk_z}=\frac{c^2 k_z}{\omega}=c\sqrt{1-\left(\frac{f_{mn}}{f}\right)^2} \tag{9}$$

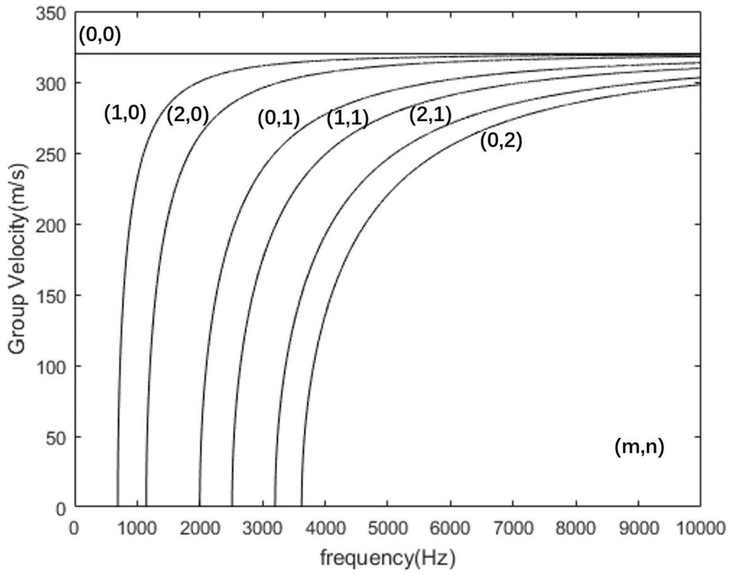

Figure 4 the curve of group velocity along with frequencies and models (only first seventh orders)

Up to now, the laser sound in the ice hole is clear. When the ice chunk hits the bottom of the ice hole, it produces a pulse signal containing a diversity of frequency components. Because of boundary conditions at the direction of $r\theta$, acoustic waves spreading through z direction shows the characteristic of dispersion. At the same model order, the higher the acoustic frequency is, the faster waves spread; at the same frequency, the higher the model order is, the more slowly waves spread. The mechanism of acoustic waves propagating leads to the "Piu" laser sound.

To verify the correctness of the physical model, the results from simulation and measurement are placed into the same time-frequency diagram. (Figure 5) The red line represents the simulation (only first seventh orders). The downward sweep yellow line at the background represents the measurement from the Twitter post above mentioned. These two kinds of lines match each other well, so the explanation of ice-hole laser sound is reasonable.

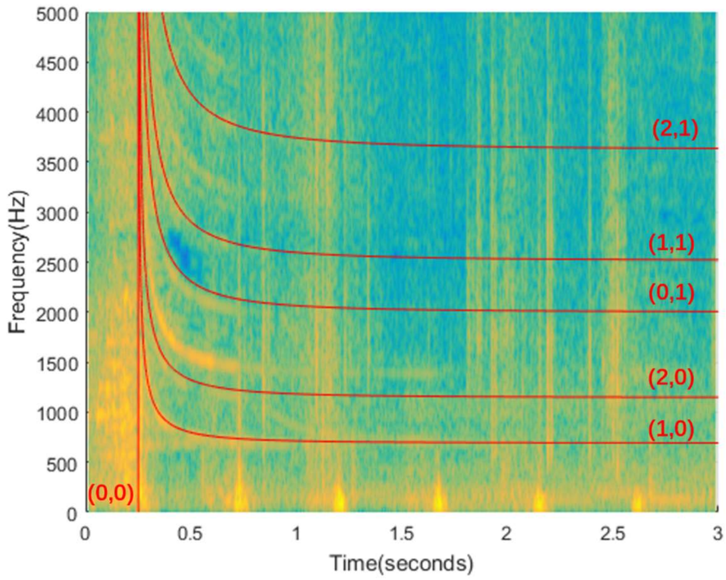

Figure 5 comparison at time-frequency diagram between simulation (only first seventh orders) and measurement

IV. DISPERSION AT SLINKY SPRING

Another kind of laser sound easily obtained is produced from slinky springs. Doctor Mix[10] shows how to use slinky spring toys to produce laser sound like what we hear in sci-fiction films. Julian Parker, etc.[21] can synthesis kinds of laser sound utilizing the dispersion mechanism and if students have difficulty getting such laser sound, they may go to Julian Parker's website[22] to get a demo, though this way is not encouraged. In this article, we focus on a physical pedagogical example.

Similarly, to know the laser sound produced by slinky springs well, a time-frequency diagram is given. (Figure 6)

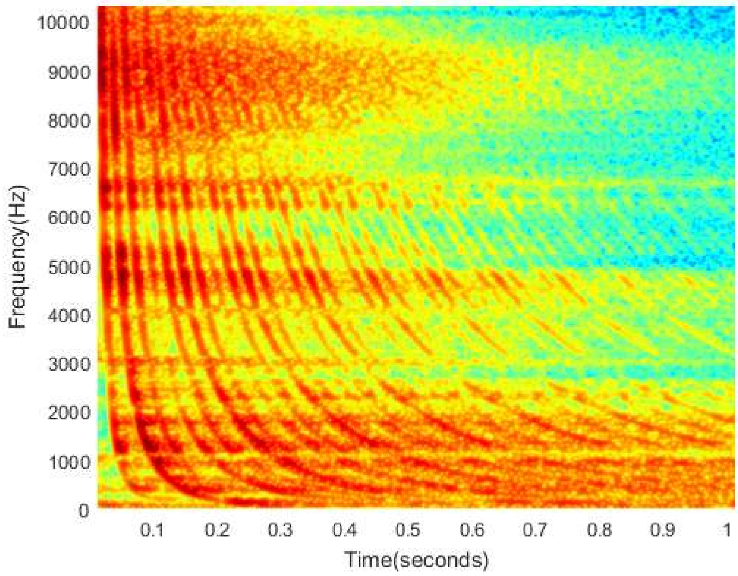

Figure 6 This is the time-frequency diagram of laser sound. Note that the sound is from Julian Parker's website[22] because their measured signal is much better than mine.

At first sight, lots of students are likely to try to explain the phenomenon using the knowledge of solid acoustics. It is because acoustic waves might show the dispersive property when they propagate in solid and interact with the boundary. However, solid acoustics is evidently more complicated than acoustics in air. For example, longitudinal waves might be coupled with transverse waves in terms of the impact of the boundary. If those students of enough competence, they may get a perfectly accurate solution. Nevertheless, as for the most undergraduates, they might be in vain if they work in that way. At this moment, lecturers had better give some clues. For example, the acoustic wavelength is at least 50mm, while the linear size of the slinky springs' cross section is no more than 2.4mm[22]. On the condition that wavelength is much larger than the size of cross section, laser sound at slinky springs is absolutely approximate to the transverse vibration of a bar.[13-15] Thanks to such an initial analysis, students are self-confidently willing to explore the mechanism of the laser sound.

It is easy to get the wave equation of transverse vibration of a bar, a fourth-order linear partial differential equation. (the detailed derivation steps are at the appendix section.)

$$\frac{\partial^4 y}{\partial x^4} = -\frac{\rho}{Q\kappa^2}\frac{\partial^2 y}{\partial t^2} \tag{10}$$

As for the equation (10), Q, ρ, κ represents Young's modulus, density, the radius of gyration respectively. And they are certain constants. Having Employed the method of separation of variables, we can get the equation of dispersion,

$$k^4 = \frac{\rho\omega^2}{Q\kappa^2} \tag{11}$$

And group velocity is

$$C_g = \frac{\partial \omega}{\partial k} = 2\sqrt{\frac{Q\kappa^2}{\rho}} \cdot k = 2\sqrt{\frac{Q\kappa^2}{\rho}}\left(\frac{\rho 4\pi f^2}{Q\kappa^2}\right)^{\frac{1}{4}} \tag{12}$$

$$C_g = \alpha \cdot \sqrt{f}, \alpha = 2\sqrt{\frac{Q\kappa^2}{\rho}}\left(\frac{4\pi\rho}{Q\kappa^2}\right)^{\frac{1}{4}} \tag{13}$$

And then we plot measurement and simulation at the same time-frequency diagram. (Figure 7)

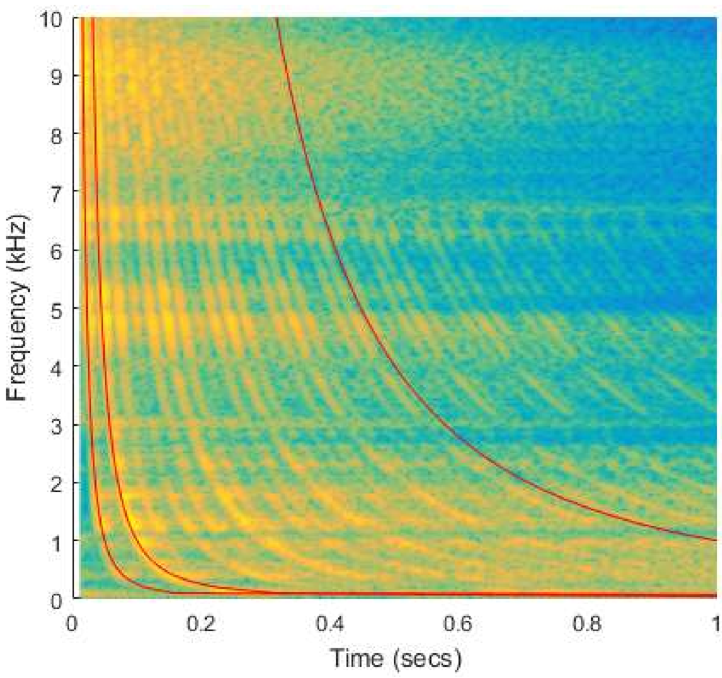

Figure 7 the time-frequency diagram of both measurement and simulation

As for measurement, there are many downward sweep lines within one second, meaning acoustic echo. In metal solid, acoustic velocity is about $10^3 m/s$ the magnitude and the slinky spring is about $20 \sim 50m$ length. When waves come across the end of the slinky spring, waves would be reflected, so within one second, there are approximately 50 times of reflections or echoes. (Figure 8) As for simulation (Figure 7), only three lines are calculated, which is enough to make the mechanism clear. Importantly, measurement and simulation match with each other well, indicating that transverse vibration of a bar at low-frequency approximation does make sense.

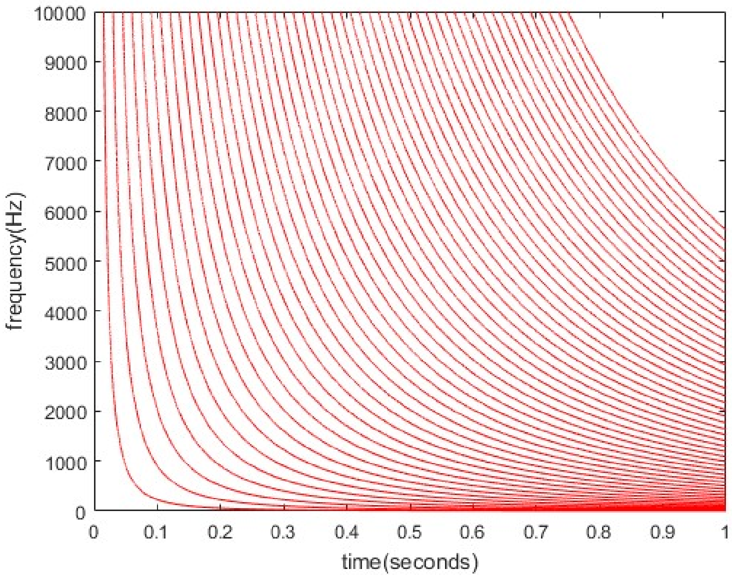

Figure 8 The amount of echo within one second is approximately 50 times.

Figure 8 shows the most fortunate scenario that the measured point is the middle of the slinky spring. In that way, the interval between two neighboring lines at the same frequency is constant. Supposing the length between the measured point and the tapping point is l_0, length of the slinky spring is l, the moment of tapping is zero point ($t=0$), we then get i echoes after t_i, in which

$$t_i \propto \frac{l_0 + (i-1)l}{\sqrt{f}} \tag{14}$$

$$t_{i+1} = \frac{l_0 + il}{\sqrt{f}} \tag{15}$$

So, the interval (Δt) between two echoes is constant in terms of the same frequency.

$$\Delta t = t_{i+1} - t_i = \frac{l_0 + il}{\sqrt{f}} - \frac{l_0 + (i+1)l}{\sqrt{f}} = \frac{l}{\sqrt{f}} \tag{16}$$

V. COMPARISON & CONCLUSION

This article mainly introduces two kinds of laser sound (whistler), laser sound in ice holes, and laser sound at slinky springs. They are very good pedagogical examples or after-school assignments. One the one hand, they are vivid and intriguing, arousing students' interests and leading them to explore proactively. On the other hand, these pedagogical examples are easily carried out, which provides much convenience to both lecturers and students during the pandemic period of COVID-19.

When we compare two kinds of laser sound, there're several same points and different points. As for the similarities, firstly, they are both laser sound. Having learned such examples, students will realize laser sound is relevant to sweep frequency signal once students meet with certain sound similar to laser sound in the future, which remarkably extend their physics intuition. Secondly, by exploring laser sound, students are bound to sharpen their skills in solving wave equations. Waves are very common and fundamental phenomena. No matter what fields students study in, physics, electronic engineering, or mechanical engineering, an in-depth understanding of wave phenomenon is beneficial for their further learning.

Moreover, students should also know two kinds of laser sound hold different points. The first difference is the medium. The ice-hold laser sound is about wave propagation in air, while laser sound at slinky springs involves wave propagation in solid, of which the point is obvious. The second difference is the degree of approximation. In terms of laser sound in ice holes, we don't do low-frequency approximation; laser sound at slinky springs needs the method of low-frequency approximation to simplify the wave equation. It is because the acoustic wavelength is roughly $0.1m$ and the diameter of the ice hole is 0.27 m , which does not satisfy the condition of low-frequency approximation; as for slinky springs, the wavelength is much larger than the linear size of the cross section of springs. If students encounter such physical details, they definitely learn how to analyze a puzzle and grasp the main issue.

VI. Appendix

the derivation of dispersion equation (8) **in the ice hole**

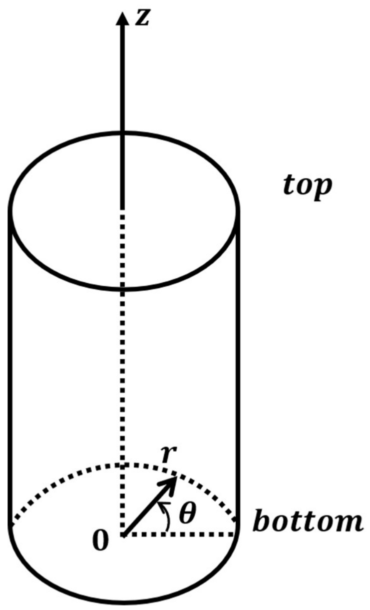

Figure 9 The cylinder hole with a rigid wall as a physical model to represent the ice hole.

The physical model is as Figure 9 shows. The depth of the ice hole is 90m ($h = 90m$), the diameter of it is 0.2718m ($2a = 0.2718m$), and the speed of sound at -20^0C is about 320m/s. The wave equation is

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{17}$$

We employ the Laplace calculator to unfold the above equation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{18}$$

We solve the equation using the method of separation of variables.

$$p = R(r)\Theta(\theta)Z(z)e^{j\omega t} \tag{19}$$

Substitute (20) into (19)

$$\begin{cases} \dfrac{d^2Z}{dz^2}+k_z^2z=0 \\ \dfrac{d^2\Theta}{d\theta^2}+m^2\Theta=0 \\ \dfrac{d^2R}{dr^2}+\dfrac{1}{r}\dfrac{dR}{dr}+\left(k_r^2-\dfrac{m^2}{r^2}\right)R=0 \end{cases} \tag{20}$$

in which,

$$k^2=\frac{\omega^2}{c^2}=k_z^2+k_r^2 \tag{21}$$

(21) is the essential dispersion equation. Afterward, we try to get the expression of k_r. In the direction of z axis, we set up a traveling wave solution.

$$\mathrm{Z}(z)=A_z e^{-jk_z z} \tag{22}$$

And at the circle, we set up a standing wave solution.

$$\Theta(\theta)=A_\theta\cos(m\theta+\varphi_m) \tag{23}$$

Note that $\Theta(\theta)$ is a periodic function with a period of 2π.

$$\Theta(\theta+2\pi)=A_\theta\cos(m(\theta+2\pi)+\varphi_m)=A_\theta\cos(m\theta+\varphi_m)=\Theta(\theta) \tag{24}$$

Thus, m must be an integer. And in the radial direction, the solution is the combination of Bessel function and Neumann function,

$$R(k_r r)=A_r J_m(k_r r)+B_r N_m(k_r r) \tag{25}$$

Because Neumann function at zero points is divergence, we only leave Bessel function.

$$R(k_r r)=A_r J_m(k_r r) \tag{26}$$

Substitute (26)(22)(23) into (19)

$$\begin{gathered} P_m=A_m J_m(k_r r)\cos(m\theta-\varphi_m)e^{j(\omega t-k_z z)}, \\ A_m=A_z A_\theta A_r \end{gathered} \tag{27}$$

Also, at the boundary of $r=a$, radial velocity equals zero.

$$v_{rm} = \frac{j}{\rho_0 \omega}\frac{\partial p_m}{\partial r} = A_m \frac{jk_r}{\rho_0 \omega}\left[\frac{dJ_m(k_r r)}{d(k_r r)}\right]\cos(m\theta - \varphi_m)e^{j(\omega t - k_z z)} \tag{28}$$

$$\left[\frac{dJ_m(k_r r)}{d(k_r r)}\right]_{r=a} = 0 \tag{29}$$

Plus, Bessel function has recurrence relations,

$$\begin{gathered}\frac{dJ_m(x)}{d(x)} = \frac{1}{2}\left[J_{m-1}(x) - J_{m+1}(x)\right] \\ \frac{dJ_0(x)}{d(x)} = -J_0(x)\end{gathered} \tag{30}$$

We employ these relations into (29) and obtain,

$$\begin{gathered}J_{m-1}(k_r a) = J_{m+1}(k_r a), (m > 0) \\ J_1(k_r a) = 0, (m = 0)\end{gathered} \tag{31}$$

Up to this step, we get the discrete value of k_r, so we use k_{mn} to represent k_r. According to Bessel function's zero points, we get the table 2. Plus, the cut-off frequency is

$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{k_{mn} \cdot c}{2\pi} = \frac{\frac{k_{mn} \cdot a}{a} \cdot c}{2\pi} = \frac{c}{2\pi a} \cdot k_{mn} a \tag{32}$$

table 2 zero points of Bessel function

$k_r a = k_{mn} a$	$m = 0$	$m = 1$	$m = 3$
$n = 0$	0	1.841	3.054
$n = 1$	3.832	5.332	6.705
$n = 2$	7.015	8.536	9.665

We substitute (32) into (21) and then we can get the dispersion equation (8).

In the following content, we continuously explore transient response. That is the Transient Green Function of the ice hole. Based on that, we may know the energy distribution at models of each order. As a result, we will hold a deeper understanding of the laser sound (whistler) of the ice hole.

Because the initial acoustic source is axisymmetric in terms of the direction of θ,

$$\frac{d\Theta}{d\theta}=0 \tag{33}$$

We can simplify the equation (27) into,

$$P_{mn}=A_{mn}J_{mn}\left(k_{mn}r\right)e^{j(\omega t-k_z z)} \tag{34}$$

According to Prof. Morse's *Theoretical Acoustics*[23] and Prof. Zhang's *Theoretical Acoustics*[24], we can obtain the expression of the Transient Green Function followingly,

$$g\left(r,z,t;r',0,0\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}g_{mn} \tag{35}$$

$$g_{mn}=\frac{cJ_m\left(k_{mn}r\right)J_m\left(k_{mn}r'\right)}{4}\int_{-\infty}^{\infty}\frac{e^{i(k_z z-\omega t)}}{-ic\pi k_z}d\omega \tag{36}$$

$$k_{mn}^2+k_z^2=\frac{\omega^2}{c^2}=k^2 \tag{37}$$

in which j and i both are imaginary units. And $i\left(k_z z-\omega t\right)=j\left(\omega t-k_z z\right)$, which is just out of using customs. In the expression (52), the factor before the sign of integration just represents the models of the standing wave at the cross section, which is decided by the shape of the ice-hold cross section. The factor does not take effect on wave propagation in the direction of z. Thus, the back part of the integration sign in the expression (52) is the key to know how the pulse signal propagates throughout the ice hole and how the energy distributes among different models.

Given

$$ck_z=c\sqrt{k^2-k_{mn}^2}=\sqrt{\omega^2-\omega_{mn}^2} \tag{38}$$

we may deduce

$$\int_{-\infty}^{\infty}\frac{e^{i(k_z z-\omega t)}}{-ic\pi k_z}d\omega=\int_{-\infty}^{\infty}\frac{e^{i\left(\sqrt{\omega^2-\omega_{mn}^2}\frac{z}{c}-\omega t\right)}}{-i\pi\sqrt{\omega^2-\omega_{mn}^2}}d\omega \tag{39}$$

If $\omega_{mn}=\omega_{00}=0$, the above formula (55) equals to

$$\int_{-\infty}^{\infty} \frac{1}{-i\pi\omega} e^{-i\omega\left(t-\frac{z}{c}\right)} d\omega = \begin{cases} 0, t < \frac{z}{c} \\ 2, t > \frac{z}{c} \end{cases} \tag{40}$$

At this point, when the acoustic source is lower than the cut-off frequency of the ice hole, there's an only plane wave propagating in the ice hole. Before the wave arrives measurement location, there's no energy at the point of measurement; once the wave arrives at the measurement location, the energy of this point increases by the form of the jump function above mentioned.

On the other hand, if $\omega_{mn} \neq \omega_{oo}$, the scenario is sort of complicated. According to the expression of point acoustic source in cylindrical coordinates[25]

$$\frac{e^{ik\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} = \frac{i}{2} \int_{-\infty}^{\infty} H_0^{(1)}(k_r r) e^{ik_z z} dk_z \tag{41}$$

and substitute k, k_z, k_r, z, r into $\frac{z}{c}, t, \sqrt{\left(\frac{z}{c}\right)^2 - t^2}, \omega, i\omega_{mn}$ respectively, we get

$$\frac{e^{i\sqrt{\omega^2-\omega_{mn}^2}\frac{z}{c}}}{i\sqrt{\omega^2-\omega_{mn}^2}} = \frac{1}{2} \int_{-\infty}^{\infty} H_0^{(1)}\left(i\omega_{mn}\sqrt{\left(\frac{z}{c}\right)^2 - t^2}\right) e^{i\omega t} dt \tag{42}$$

$$\frac{2e^{i\sqrt{\omega^2-\omega_{mn}^2}\cdot\frac{z}{c}-2i\omega t}}{i\sqrt{\omega^2-\omega_{mn}^2}} = \int_{-\infty}^{\infty} H_0^{(1)}\left(i\omega\sqrt{\left(\frac{z}{c}\right)^2 - t^2}\right) e^{-i\omega t} dt \tag{43}$$

Evidently, formula (55) is a kind of Fourier Transformation, in which the expression in the time domain is $H_0^{(1)}\left(i\omega_{mn}\sqrt{\left(\frac{z}{c}\right)^2 - t^2}\right)$ and the expression in the frequency domain is $\frac{2e^{i\sqrt{\omega^2-\omega_{mn}^2}\frac{z}{c}-2i\omega t}}{i\sqrt{\omega^2-\omega_{mn}^2}}$. At the same time, we can also get the Fourier Inverse Transformation,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2e^{i\sqrt{\omega^2-\omega_{mn}^2}\frac{z}{c}-2i\omega t}}{i\sqrt{\omega^2-\omega_{mn}^2}}e^{i\omega t}d\omega = H_0^{(1)}\left(i\omega_{mn}\sqrt{\left(\frac{z}{c}\right)^2-t^2}\right) \tag{44}$$

$$\int_{-\infty}^{\infty}\frac{e^{i\sqrt{\omega^2-\omega_{mn}^2}\frac{z}{c}-i\omega t}}{-i\sqrt{\omega^2-\omega_{mn}^2}}d\omega = -\pi H_0^{(1)} i\omega_{mn}\sqrt{\left(\frac{z}{c}\right)^2-t^2} \tag{45}$$

Up to this step, formula (45) is exactly the result of the equation (55). Followingly, we try to analyze the property of formula (45) which is divided into two cases. On the one hand, if $0 < t < \frac{z}{c}$, $-\pi H_0^{(1)}\left(i\omega_{mn}\sqrt{\left(\frac{z}{c}\right)^2-t^2}\right)$ only has an imaginary part. Namely, the real part of $\int_{-\infty}^{\infty}\frac{e^{i\sqrt{\omega^2-\omega_{mn}^2}\frac{z}{c}-i\omega t}}{-i\omega\sqrt{\omega^2-\omega_{mn}^2}}d\omega$ is zero. On the other hand, if $t > \frac{z}{c}$, the real part of $\int_{-\infty}^{\infty}\frac{e^{i\sqrt{\omega^2-\omega_{mn}^2}\frac{z}{c}-i\omega t}}{-i\omega\sqrt{\omega^2-\omega_{mn}^2}}d\omega$ is

$$J_0\left(\omega_{mn}\sqrt{t^2-\left(\frac{z}{c}\right)^2}\right) \tag{46}$$

To figure out formula (46), we make use of the asymptotic expression of Bessel Functions.

$$J_0\left(\omega_{mn}\sqrt{t^2-\left(\frac{z}{c}\right)^2}\right) \approx \sqrt{\frac{2}{\pi\omega_{mn}t}}\cos\left(\omega_{mn}\sqrt{t^2-\left(\frac{z}{c}\right)^2}-\frac{\pi}{4}\right) \tag{47}$$

By the asymptotic expression of Bessel Functions, the amplitude is decreasing along with the augment of cut-off frequency and time. (Figure 10)

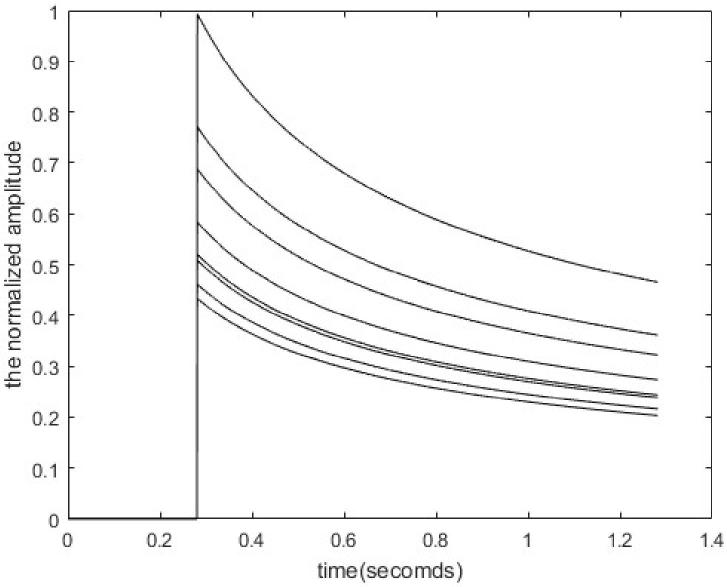

Figure 10 This is the result that shows the relative vibrating amplitude of first eight order models. From top to bottom, the order number is (2,2), (1,2), (0,2), (2,1), (1,1), (0,1), (2,0), (1,0) respectively.

Meanwhile, we can know how transient frequency changes.

$$\phi = \omega_{mn}\sqrt{t^2 - \frac{z^2}{c^2}} - \frac{\pi}{4}$$
$$\frac{\partial \phi}{\partial t} = \frac{\omega_{mn} t}{\sqrt{t^2 - \frac{z^2}{c^2}}} = \frac{\omega_{mn}}{\sqrt{1 - \frac{z^2}{c^2 t^2}}} \tag{48}$$

That is, the transient frequency goes downward with the time lapsing, as Figure 11 shows.

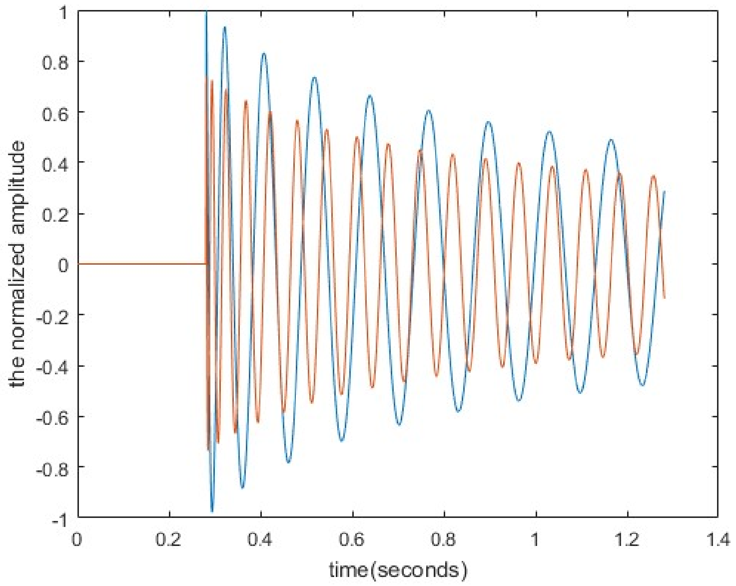

Figure 11 The blue line represents the (0,1) model and the red one represents the (0,2) model. This is an only schematic diagram. It is because vibrating frequencies (carrier frequency) of both two models are reduced by 20 times so that we can grasp vibrating frequency is decreasing within 1.2 seconds.

Up to this point, we have worked out the transient response at the direction of z or Green Function of the ice hole. To conclude, formula (21) indicating dispersion claims the propagation velocity of different frequencies and models; formula (35)(36) indicating ice-hole Green function claims the propagation energy of different frequencies and models. Both of them lead to the production of laser sound or whistler. Note that undergraduates are more likely to feel difficult to the detailed derivation of dispersion equation and Green Function of the ice hole, but lecturers or graduates should keep the process clear in their minds.

the derivation of dispersion equation (11) **at slinky springs**

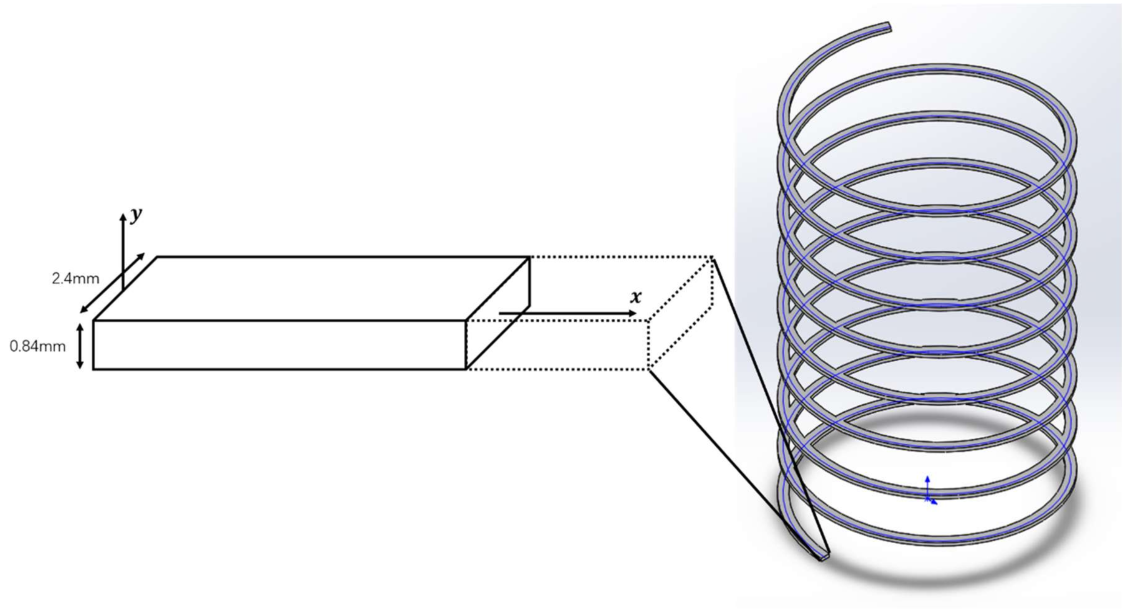

Figure 12 The vibration of a bar illustrates the laser sound at slinky springs. The right one is one section of slinky spring drawn by SolidWorks, of which its cross section is a rectangle shape with a length of 2.4mm and width of 0.84mm.

Prof. Morse[26] in his *Theoretical Acoustics* pointed out that transverse vibration of a bar at low frequency is a fourth linear partial differential equation,

$$\frac{\partial^4 y}{\partial x^4} = -\frac{\rho}{Q\kappa^2}\frac{\partial^2 y}{\partial t^2} \tag{49}$$

in which Q means Young's modulus and ρ means density. κ means the radius of gyration. If the cross-sessional shape is a rectangle,

$$\kappa = \left(\frac{a}{\sqrt{12}}\right) \tag{50}$$

in which a is width perpendicular to the centerline. Similarly, we use the method of separation of variables to absolve the wave equation (49). We set up

$$y = Y(x)e^{-i\omega t} \tag{51}$$

and substitute (51) into (49)

$$\frac{d^4 Y}{dx^4} = \frac{\rho\omega^2}{Q\kappa^2}Y \tag{52}$$

$$Y(x) = d_1 \cosh\left[\left(\frac{\rho\omega^2}{Q\kappa^2}\right)^{\frac{1}{4}} x\right] + d_2 \sinh\left[\left(\frac{\rho\omega^2}{Q\kappa^2}\right)^{\frac{1}{4}} x\right] + d_3 \cos\left[\left(\frac{\rho\omega^2}{Q\kappa^2}\right)^{\frac{1}{4}} x\right] + d_4 \sin\left[\left(\frac{\rho\omega^2}{Q\kappa^2}\right)^{\frac{1}{4}} x\right] \tag{53}$$

At this point, we define the wave beam $k = \left(\frac{\rho\omega^2}{Q\kappa^2}\right)^{\frac{1}{4}}$.

$$y = \left[d_1 \cosh(kx) + d_2 \sinh(kx) + d_3 \cos(kx) + d_4 \sin(kx)\right] e^{-i\omega t} \tag{54}$$

Therefore, we obtain our target dispersion equation,

$$k^4 = \frac{\rho\omega^2}{Q\kappa^2} \tag{55}$$

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