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Scale factors for centrifuge modeling of unsaturated flow

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ABSTRACT: Although scaling laws for centrifuge modeling of unsaturated water flow have been investigated in the past, some controversial issues still remain unclear. In order to provide insight on the mechanics governing the flow of water under an increased gravitational field, a consistent framework for centrifuge modeling of saturated and unsaturated water flow is presented in this paper. To this effect, the solution of the governing equations for the model and prototype are analyzed in order to infer the flow scale factors. Specifically, the governing equations for centrifuge flow in a permeameter are deduced and compared to the vertical $1g$ case. The unsaturated flow rate in the model is found to be properly scaled by $1/N$ and time is scaled by N^2, where N is the ratio between the centrifuge and the gravitational accelerations.

1 INTRODUCTION

Working with an inherently variable material, the geotechnical engineer usually estimates the margin of safety of geotechnical structures based only on analytic soil-prototype behavioral models. Sometimes, the uncertainties involved in the analysis prompt the engineer to search physical methods for prediction the prototype's behavior. Centrifuge modeling represents a feasible alternative method since the stress level developed within the model equals that in the prototype, the cost of testing is comparatively small, and the long-term behavior of the geotechnical model can be predicted in reduced time frames.

The main objectives of centrifuge testing are the investigation of the behavior of prototypes, the investigation of new phenomena, the parametric study of uncommon scenarios, and the validation of numerical methods. Measurement of soil hydraulic conductivity has also been conducted using a centrifuge facility as a laboratory tool. In this case, high hydraulic gradients can be induced and a homogeneous stress field can be induced in soil specimens. Ko (1988) notes that the inference of prototype behavior has been the most common application of centrifuge testing. This technique is normally referred as prototype modeling and it is usually accurate in situations controlled by body forces. Assessment of analytic tools (e.g. limit equilibrium) using centrifuge modeling has been useful for validation purposes (Zornberg et al. 1997).

Centrifuge flow modeling has received increased interest, particularly regarding the assessment of unsaturated flow. Therefore, the main objective of the paper is to present a consistent framework of the theory of saturated and unsaturated water flow under a centrifugal field. When compared with a vertical $1g$ test, the framework is suitable for deduction of the scale factors that govern the relationship between a prototype and a reduced model.

2 BACKGROUND

Basic concepts regarding unsaturated soil behavior and centrifuge modeling are reviewed in this section. Coupling of centrifuge modeling concepts and unsaturated soil properties evaluation is then provided to evaluate the laws governing the centrifuge modeling of unsaturated water flow.

2.1 *Unsaturated flow concepts*

Fredlund & Rahardjo (1993) describe the unsaturated soil as a four-phase material in which the meniscus surface acts as a fourth phase. The meniscus surface phase induces tensile forces that contribute to soil inter-particles attraction.

The effect of suction on the soil shear strength are controlled by the total suction (ψ) of the soil mass. The total suction can be defined as the sum of the matric suction (related to the meniscus effect) and the osmotic suction. The matric suction plays an important role in situations where the soil degree of saturation can vary widely as the case of unsaturated water flow. Since matric suction is related to body forces, its magnitude will be affected when a model is accelerated in a centrifuge apparatus. The osmotic suction, on the other way, is related to the ion concentration in the soil mass. It is related to diffusion and adsorption processes that occur among ions, water molecules, and soil particles. Osmotic processes are independent of the body force and, consequently, their magnitude will not be affected when a model is accelerated in a centrifuge facility. For the purpose of this study, the magnitude of osmotic suction will be considerably negligible.

The soil–matric suction relationship is defined by the characteristic curve of the soil. The relationship between unsaturated hydraulic conductivity and suction is commonly defined by the hydraulic conductivity function of the soil. The characteristic curve and the hydraulic conductivity function typically show hysteresis (i.e. different drying and wetting paths). Hysteresis has been explained by different air flow patterns during these two processes. The hydraulic conductivity function has been described by mathematical models, such as those proposed by Mualem (1976), Van Genuchen (1981) and Fredlund & Xing (1994).

2.2 *Centrifuge modeling concepts*

Scale factors relating model and prototype must be defined in order to infer the response of prototypes based on the monitored response of reduced scale models. The model should be similar in geometric, kinematic, and dynamic aspects. Similitude aspects have been generally defined by Buckingham's Π theorem or by inspection of the governing equations of the phenomenon considered (Cargill & Ko 1983). Scale factors that have been typically adopted for centrifuge modeling are given in Table 1 (Ko 1988). They have been derived based on the assumption that both prototype and model have the same material characteristics.

It is important to note that, depending on the phenomenon considered, the time scale factor can differ. The differences can be explained by the different differential equations that govern the various phenomena. For example, the time scale factor for consolidation (based on the diffusion equation) is different from the time scaling factor for steady-state flow (based on Laplace's equation).

Validation of the scale factors has been typically made by direct comparison between prototype and model responses, or, in the absence of monitored prototype, by adopting the modeling of models technique (Ko 1988).

Regarding flow modeling, the scale factors in Table 1 are for the case in which the soil sample does not show volumetric changes in response to stress level increases. If volumetric changes occur, the soil hydraulic conductivity will change during the flow process, invalidating the scale factors (i.e. the model's hydraulic conductivity will be different from the prototype's).

2.3 *Discharge velocity under 1 g gravity level*

Figure 1 shows a schematic representation of 1D flow through a control volume under vertical 1 g and

Table 1. Scale factors for centrifuge modeling (adapted from Ko 1988).

Quantity	Prototype/model
Length	N
Area	N^2
Volume	N^3
Velocity	1
Acceleration	$1/N$
Mass	N^3
Force	N^2
Energy	N^3
Stress	1
Strain	1
Mass density	1
Energy density	1
Time (dynamic)	N
Time (creep)	1
Time (diffusion)	N^2
Frequency	$1/N$
Pressure	1
Flow velocity	$1/N$
Flow quantity	N
Head	N
Capillary rise	N

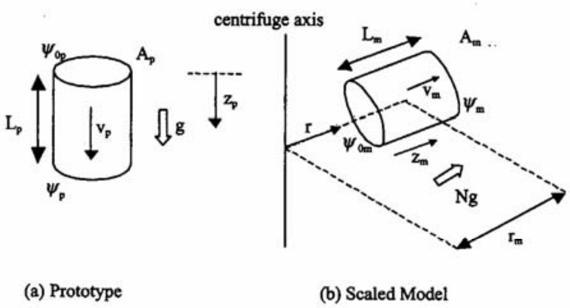

Figure 1. Flow patterns for vertical 1 g and centrifugal flow.

centrifugal acceleration. In both cases, the flow pattern is governed by a flow potential (Φ) i.e. the energy per unit mass of fluid. The flow potential, described by Bernoulli's equation, for the 1 g case is defined by (Cherry & Freeze 1979):

$$\Phi = gz + \frac{v_s^2}{2} + \frac{p}{\rho_w} \tag{1}$$

where g denotes the gravity acceleration, z the elevation, v_s the seepage velocity, p the pressure and ρ_w the fluid density.

The first two terms on Bernoulli's equation are the potential and the kinetic energy of the fluid. The third term corresponds to the work done on the fluid by pressure variations. Since the seepage velocity within the soil mass is comparatively low, the second term is generally disregarded. The relationship among the total head (h), the elevation head (z) and the pressure heads (h_p) is obtained dividing Equation (1) by g. Combining the total head relationship with Darcy's equation (assumed valid under unsaturated conditions) and considering that suction equals $-\gamma_w h_p$, the discharge velocity is expressed by

$$v_p = -k(\psi)\left[1 - \frac{1}{\gamma_w}\frac{\partial\psi}{\partial z_p}\right] \quad (2)$$

where v_p is the discharge velocity in the prototype, $k(\psi)$ is the hydraulic conductivity of the soil for a total suction ψ.

2.4 *Discharge velocity under a centrifugal field*

The gravitational field in a centrifuge is related to the centrifugal acceleration (a_c) as follows:

$$a_c = \omega^2 r_m = Ng \quad (3)$$

where ω and r_m are the angular velocity and the average radial distance. Considering the control volume in Figure 1b, it can be observed that the induced gravitational field varies along the radial flow path of an incompressible fluid. The flow potential in this case is given by

$$\Phi = \frac{1}{2}\omega^2 r_m^2 - \frac{v_s^2}{2} + \frac{p}{\rho_w} \quad (4)$$

The flow potential stated by Equation (4) is the Bernoulli's equation for a centrifugal field. Also in this case, the seepage velocity can be disregarded. However, this implies that no turbulent flow occurs during centrifuge testing. In other words, the Reynolds' numbers of prototype and model should be similar.

Expressing Darcy's law in terms of flow potential, the discharge velocity is given by

$$v_m = -\frac{k(\psi)}{\gamma_w}\left(\rho_w\omega^2 r_m - \frac{\partial\psi}{\partial z_m}\right) \quad (5)$$

where v_m is the discharge velocity in the model, γ_w is the water unit weight and ψ is the total suction. The term $\partial\psi/\partial z_m$ is the suction gradient acting through the soil model. This formulation is similar to that proposed by Nimmo et al. (1987, 1992) and Conca & Wright (1990) to describe the flow density in an unsaturated soil sample placed in a centrifuge.

3 FLOW GOVERNING EQUATIONS

Governing equations for flow in soils describing the water or pollutants propagations, under saturated or unsaturated conditions, are based on the principle of continuity. Considering the control volume shown in Figure 1a and assuming a constant volume of solids through the flow process and an incompressible fluid, the difference between the inflow and outflow equals the water stored within the soil mass. Assuming the validity of Darcy's law, the continuity principle for $1g$ vertical flow can be expressed as

$$\frac{\partial}{\partial z_p}\left[k\left(z_p\right)\frac{\partial h}{\partial z_p}\right] = \left(\frac{1}{1+e}\right)\frac{\partial}{\partial t}[Se] \quad (6)$$

where e is the void ratio, S is degree of saturation and $k\left(z_p\right)$ is the hydraulic conductivity in the z_p direction. Equation (6) describes flow under steady-state or transient conditions in one direction.

3.1 *Laplace's equation*

Assuming a homogeneous, isotropic, saturated medium, the hydraulic conductivity becomes constant and can be placed outside the derivative in Equation (6). Also, assuming no volume changes, the derivatives $\partial e/\partial t$ and $\partial S/\partial t$ equal zero. These hypotheses characterize the steady-state saturated seepage. In this case, the continuity principle is then expressed as

$$\frac{\partial^2 h}{\partial z_p^2} = 0 \quad (7)$$

which is Laplace's equation. For the specific case of a permeameter, as depicted in Figure 1a, the solution in terms of suction is

$$\psi(z_p) = \gamma_w z_p - \frac{\gamma_w v_p}{k} z_p + \psi_{0p} \quad (8)$$

where ψ_{0p} is the suction at the upper boundary in the permeameter, and γ_w is the water unit weight at $1g$.

3.2 *Richards' equation*

Richards (1931) developed a framework for water flow in unsaturated soils. The main assumptions are that the air phase is free to flow through the soil, the hydraulic conductivity can be described by the suction, Darcy's law is valid, and the material is isotropic. Richards' equation for unsaturated steady-state flow can be expressed as

$$\frac{\partial^2 h_p}{\partial z_p^2} = v_p \frac{\partial[1/k(\psi)]}{\partial\psi}\frac{\partial\psi}{\partial z_p} \quad (9)$$

The solution of Equation (9) for the permeameter shown in Figure 1a is

$$\psi(z_p) = \gamma_w z_p - \gamma_w v_p \cdot \Theta_p(\psi, z_p) + \psi_{0p} \quad (10)$$

where the term $\Theta_p(\psi, z_p)$ equals

$$\Theta_p(\psi, z_p) = \int_{z_{0p}}^{z_p} \frac{1}{k(\psi)} \mathrm{d}z_p \quad (11)$$

Considering the case of zero flow rate Equation (10) equals

$$\psi(z_p) = \gamma_w z_p + \psi_{0p} \quad (12)$$

3.3 *Governing equation for saturated steady-state centrifugal flow*

The governing equation for a centrifugal flow can be deduced by using the Darcy's law (Eq. (5)), the principle of continuity and assuming a saturated homogeneous and isotropic medium. In this case

$$\frac{\partial^2 \psi}{\partial z_m^2} = \rho_w \omega^2 \tag{13}$$

Considering the permeameter shown in Figure 1b at steady state, the solution is given by

$$\psi(z_m) = \rho_w \omega^2 r_m z_m - \frac{v_m \cdot \gamma_w}{k} z_m + \psi_{0m} \tag{14}$$

Differently than for Laplace's equation solution, Equation (14) shows that the suction pattern in a centrifuge is controlled by a function of the angular velocity and the average radial distance.

3.4 *Governing equation for unsaturated steady-state centrifugal flow*

The governing equation for unsaturated centrifugal flow is based on the same assumptions as Richards' equation. Steady state is reached during centrifugation if the following two conditions are met. First, the sample volume must be constant through the process (i.e. $\partial e(\psi)/\partial t = 0$). This implies that the soil sample does not swell or contract under an induced gravitational field. Second, the suction profile becomes constant through the soil sample. The governing equation can be expressed as

$$\frac{\partial^2 \psi}{\partial z_m^2} = v_m \cdot \gamma_w \frac{\partial[1/k(\psi)]}{\partial \psi} \frac{\partial \psi}{\partial z_m} + \rho_w \omega^2 \tag{15}$$

Similar to the solution of Richards' equation, the solution of Equation (15) involves an iterative process since the hydraulic conductivity is a function of the suction. For the permeameter in Figure 1b and considering the average radial distance (r_m) the solution can be expressed as

$$\psi(z_m) = \rho_w \omega^2 r_m z_m - \gamma_w v_m \cdot \Theta_m(\psi, z_m) + \psi_{0m} \tag{16}$$

where the term $\Theta_m(\psi, z_m)$ is the integral of the inverse of the hydraulic conductivity function along model's length, and $\psi_{0m}$is the suction on the top boundary. For zero flow rate, Equation (16) becomes

$$\psi(z_m) = \rho_w \omega^2 r_m z_m + \psi_{0m} \tag{17}$$

The suction profile through the specimen can then be obtained using the characteristic curve and hydraulic conductivity function of soil sampled and the imposed boundary conditions.

4 SCALE FACTORS

The study of the unsaturated flow pattern in scaled models is one of the main objectives of centrifuge modeling. In this case, the purpose is the inference about flow patterns on a 1-, 2- or 3D scaled model, in order to provide some insights about a prototype behavior (Cargill & Ko 1983).

Cargill & Ko (1983) and Butterfield (2001) present the deduction of the scale factors for water flow based on dimensional analysis. They concluded that flow rate is scaled by N and time is scaled by $1/N^2$. Cargill & Ko (1983) also indicate that the scale factor for capillary height is $1/N$.

Lord (1999) and Depountis et al. (2001) deduce the scale factor for capillary head based on Poiseuille's model for capillary flow. Lord (1999) shows theoretically that the capillary head is scaled by $1/N$. Lord (1999) also presents the scale factor for time as equal to $1/N^2$. Depountis et al. (2001) observe that the scale factors deduced based on Poiseuille's model, scales correctly the prototype behavior under a certain range of g-levels.

Barry et al. (2001) by inspection of governing equation reached similar scale factors as presented by Cargill & Ko (1983).

The framework presented herein evaluates the solution of the governing equation for a permeameter under $1g$ (prototype) and under centrifugal acceleration (model). The model's governing equation solution must contain implicitly the scale factors that exist in respect to prototype. Therefore, considering the length scale between model and prototype given by

$$z_p = N z_m \tag{18}$$

and the respective boundary conditions in the model's governing equation solution, the scale factors can be obtained.

4.1 *Suction scale factor*

The centrifuge test of unsaturated water flow is carried out in two steps. The first step consists in the acceleration of the model until the desired g-level without any flow. The second step is characterized by the water flow until steady state is reached. Therefore, Equation (17) describes the suction initial state at the end of the first step. Considering Equations (3), (18), and (17):

$$\psi(z_m) = \gamma_w z_p + \psi_{0m} \tag{19}$$

Comparing Equations (19) and (12), it can be observed that the suction profile in the prototype is the same as that in the model under the condition of no flow if the same boundary condition of prototype is imposed to the model ($\psi_{0m} = \psi_{0p}$). That is,

$$\psi(z_m) = \psi(z_p) \tag{20}$$

That is, the suction scale factor equals 1.

If prototype and model are constituted of same material, $\Theta_m(\psi, z_m)$ scales by $1/N$ with respect to $\Theta_p(\psi, z_p)$(Dell'Avanzi & Zornberg 2001). Therefore, observing the scale factor for discharge velocity as deduced ahead, and assuming prototype and model constituted by the same material, Equation (20) is also valid.

4.2 *Unsaturated discharge velocity scale factor*

From Equation (5) and considering the scaling for length and suction stated in Equations (18) and (20), the discharge velocity in the model can be expressed as

$$v_m = -\frac{k(\psi)}{\gamma_w}\left(\rho_w\omega^2 r_m - N\frac{\partial\psi}{\partial z_p}\right) \tag{20a}$$

Using Equation (3) and considering $\gamma_w = \rho_w g$:

$$v_m = -k(\psi)\cdot N\left(1 - \frac{1}{\gamma_w}\frac{\partial\psi}{\partial z_p}\right) \tag{21}$$

Comparing Equations (2) and (21), it can be seen that:

$$v_m = Nv_p \tag{22}$$

which shows that for unsaturated flow conditions the model discharge velocity scales by N with respect to the prototype discharge velocity. The result agrees with Barry et al. (2001) for the case in which model and prototype have the same material and same fluid density.

4.3 *Unsaturated flow rate scale factor*

The model flow rate is given by

$$Q_m = v_m A_m \tag{23}$$

where A_m is the model's cross section area (see Fig. 1b). Observing that discharges velocities scale by N (Eq. (18)) and areas scale by N^2 (see Table 1), the flow rate scale factor is given by

$$\begin{aligned} Q_m &= Nv_p\frac{1}{N^2}A_p \\ Q_m &= \frac{Q_p}{N} \end{aligned} \tag{24}$$

which shows that for unsaturated flow conditions the model flow rate scales by $1/N$ with respect to the prototype flow rate. The relationship presented by Equation (20) is the same scale factor proposed by Barry et al. (2001) for prototype and model composed by same material.

Table 2. Scale factors for unsaturated water flow.

Quantity	Prototype/model
Unsaturated flow rate	N
Unsaturated discharge velocity	$1/N$
Time (unsaturated flow)	$1/N^2$

4.4 *Time scale factor*

The time required for a tracer (or a water particle) to travel from top to bottom of model is given by

$$t_m = \frac{nL_m}{v_m} \tag{25}$$

where t_m is the model's time lag and n is the soil's porosity. If model and prototype have same material, the porosities will be the same. Therefore, considering the scale factor for discharge velocities (Eq. (18)) and the relationship between model and prototype lengths $(L_p = NL_m)$, Equation (23) becomes equal to

$$t_m = \frac{1}{N^2}\frac{nL_p}{v_p} \tag{26}$$

Since the travel time in the prototype is defined by $t_p = nL_p/v_p$, the time scale factor is

$$t_m = \frac{1}{N^2}t_p \tag{27}$$

which agrees with Lord (1999) and Barry et al. (2001).

A summary of the scale factors for unsaturated flow parameters is presented in Table 2. These scale factors are consistent with the results obtained by dimensional analysis (Cargill & Ko 1983, Butterfield 2000) and by inspection of governing equation (Lord 1999, Depountis et al. 2001, Barry et al. 2001).

5 CONCLUSIONS

A consistent framework was developed in order to compare $1g$ vertical flow to Ng centrifugal flow under saturated and unsaturated conditions. The governing equations concerning saturated and unsaturated water flow were deduced based on the principle of continuity. While the saturated flow under $1g$ conditions is governed by Laplace's equation, the saturated flow under centrifuge conditions is governed by a differential equation which is a function of the angular acceleration. Also, since the gravitational field varies along the sample in a centrifuge test, different solutions were obtained for the suction patterns concerning $1g$ and Ng-level tests respectively. The suction pattern in a $1g$ environment varies linearly whereas in an Ng-level varies quadratic.

Scale factors between model and prototype were derived based on the assumption that the solution of the governing equations for a permeameter in a centrifuge apparatus can be used to evaluate the behavior of a prototype.

The framework discussed herein was developed based on three principal assumptions. First, prototype and model have the same characteristic curve and same hydraulic conductivity function. Second the characteristic curve and hydraulic conductivity functions do not change when the g-level is increased. Third, Darcy's law is valid under increased g-level. The scale factors were deduced by solving the governing equation and substitution of the model-prototype lengths ratio in the respective governing equations. It can be concluded that the unsaturated flow rate scales by $1/N$, the discharge velocity scales by N and time scales by $1/N^2$.

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